

# ON SOME DEGENERATE NON-LOCAL PARABOLIC EQUATION ASSOCIATED WITH THE FRACTIONAL $p$ -LAPLACIAN

CIPRIAN G. GAL AND MAHAMADI WARMA

ABSTRACT. Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary bounded open set. We consider a degenerate parabolic equation associated to the fractional  $p$ -Laplace operator  $(-\Delta)_p^s$  ( $p \geq 2$ ,  $s \in (0, 1)$ ) with the Dirichlet boundary condition and a monotone perturbation growing like  $|\tau|^{q-2}\tau$ ,  $q > p$  and with bad sign at infinity as  $|\tau| \rightarrow \infty$ . We show the existence of locally-defined strong solutions to the problem with any initial condition  $u_0 \in L^r(\Omega)$  where  $r \geq 2$  satisfies  $r > N(q-p)/sp$ . Then, we prove that finite time blow-up is possible for these problems in the range of parameters provided for  $r, p, q$  and the initial datum  $u_0$ .

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2010 *Mathematics Subject Classification.* 35R11, 35K55, 35K65.

*Key words and phrases.* Fractional  $p$ -Laplace operator, Dirichlet boundary conditions, degenerate non-linear parabolic equations, existence and regularity of local solutions, blow up.

The work of the second author is partially supported by the Air Force Office of Scientific Research under the Award No. FA9550-15-1-0027.

## 1. INTRODUCTION

The article is concerned with the following non-local initial-boundary value problem for the degenerate parabolic equation

$$\begin{cases} \partial_t u(x, t) + (-\Delta)_p^s u(x, t) - |u(x, t)|^{q-2} u(x, t) = f(x, t) & (x, t) \in \Omega \times (0, T) \\ u(x, t) = 0 & (x, t) \in (\mathbb{R}^N \setminus \Omega) \times (0, T) \\ u(x, 0) = u_0(x) & x \in \Omega. \end{cases} \quad (1.1)$$

Here  $u_0 \in L^r(\Omega)$ ,  $2 \leq p, q, r < \infty$ ,  $T > 0$ ,  $f$  is a given function,  $(-\Delta)_p^s$  denotes the fractional  $p$ -Laplace operator and  $\Omega$  is an arbitrary bounded open subset of  $\mathbb{R}^N$ ,  $N \geq 1$ . To introduce the fractional  $p$ -Laplace operator, let  $0 < s < 1$ ,  $p \in (1, \infty)$  and set

$$\mathcal{L}^{p-1}(\mathbb{R}^N) := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable, } \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}} dx < \infty \right\}.$$

For  $u \in \mathcal{L}^{p-1}(\mathbb{R}^N)$ ,  $x \in \mathbb{R}^N$  and  $\varepsilon > 0$ , we let

$$(-\Delta)_{p,\varepsilon}^s u(x) = C_{N,p,s} \int_{\{y \in \mathbb{R}^N, |y-x| > \varepsilon\}} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+ps}} dy,$$

where the normalized constant

$$C_{N,p,s} = \frac{s 2^{2s} \Gamma\left(\frac{ps+p+N-2}{2}\right)}{\pi^{\frac{N}{2}} \Gamma(1-s)}$$

and  $\Gamma$  is the usual Gamma function (see, e.g., [5, 8, 9, 10, 11] for the linear case  $p = 2$ , and [24, 25] for the general case  $p \in (1, \infty)$ ). The fractional  $p$ -Laplacian  $(-\Delta)_p^s$  is defined by the formula

$$\begin{aligned} (-\Delta)_p^s u(x) &= C_{N,p,s} \text{P.V.} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{u(x) - u(y)}{|x-y|^{N+ps}} dy \\ &= \lim_{\varepsilon \downarrow 0} (-\Delta)_{p,\varepsilon}^s u(x), \quad x \in \mathbb{R}^N, \end{aligned} \quad (1.2)$$

provided that the limit exists. We notice that if  $0 < s < (p-1)/p$  and  $u$  is smooth (i.e., at least bounded and Lipschitz continuous), then the integral in (1.2) is in fact not really singular near  $x$ .

The case  $p = 2$  and  $f \equiv 0$ , which corresponds to the case of a semilinear fractional heat equation, sufficient conditions for the existence of weak solutions with  $u_0 \in L^2(\Omega)$ , and strong solutions for  $u_0 \in L^\infty(\Omega)$ , have already been proved in [13]. Additionally, further dynamical properties (i.e., existence of finite dimensional global attractors and global asymptotic stabilization to steady states as time goes to infinity) were also derived for a semilinear parabolic problem of the form

$$\partial_t u + (-\Delta)_2^s u + h(u) = 0 \quad \text{in } \Omega \times (0, \infty), \quad u = 0 \quad \text{on } (\mathbb{R}^N \setminus \Omega) \times (0, \infty), \quad (1.3)$$

with nonlinearity  $h(\tau)$  which has a good sign at infinity as  $|\tau| \rightarrow \infty$ , and which is coercive in a precise sense. Finally, some blow-up results were also proved in [13] for (1.3) with  $h(\tau) \sim -|\tau|^{q-2} \tau$ , as  $|\tau| \rightarrow \infty$ , emphasizing the same critical blow-up exponent  $q = p = 2$  as for the corresponding parabolic equation associated with the classical Laplace operator  $-\Delta$ . We extend our work of [13] to prove the local in time existence of solutions to parabolic equations with degenerate fractional diffusion and more singular kernels using an approach based on [3, 4] and also developed further

in [2]. Although our general scheme follows closely that of [2, 3, 4], many of the key lemmas used in the case of the classical  $p$ -Laplace operator cannot be adapted or exploited in their classical form to deal with the fractional  $p$ -Laplacian  $(-\Delta)_p^s$  for  $s \in (0, 1)$  and  $p \in (1, \infty)$ . Hence, we develop some new techniques including some new functional inequalities allowing us to extend the results of [2] in the present setting. Among these new tools that we derive it is worth mentioning a *nonlinear version* of the classical Stroock-Varopoulos inequality (see Lemma 3.9) which is an important inequality in the theory of Markovian semigroups, and a new coercitivity estimate (see Lemma 3.10) which is also crucial in the proofs for the energy estimates. In particular, Lemma 3.9 extends the classical Stroock-Varopoulos inequality which was available only in the case  $p = 2$  (see [20, 21]) and covers also the case when  $p \neq 2$ . Lemma 3.9 is the main tool in proving our first main result of Theorem 2.3. Then we also generalize the blow-up results of [13] to the present case (see Theorem 2.5) when  $q > p$  following a technique adapted from [18]. We emphasize that our results hold without any regularity assumptions on  $\Omega$ . There is vast literature on degenerate parabolic equations involving the classical diffusion operator  $-\Delta_p$ . We refer the reader to the following list [2, 3, 4, 6, 18] (and references contained therein) which is not meant to be exhaustive.

To the best of our knowledge, little is known about parabolic problems associated with the fractional  $p$ -Laplacian  $(-\Delta)_p^s$  with the exception of [22, 24, 25]. In [25], some regularity results are provided for the quasi-linear parabolic equation  $\partial_t u + (-\Delta)_p^s u = 0$  and Dirichlet boundary condition  $u = 0$  in  $\mathbb{R}^N \setminus \Omega$ , whereas in [22] for the same quasi-linear problem, it is proven the eventual boundedness of  $u$  in  $L^\infty((\tau, T); L^\infty(\Omega))$ , for every  $\tau > 0$ , provided that the initial datum  $u_0 \in L^p(\Omega)$ . Most recently an integration by parts formula for the regional fractional  $p$ -Laplace operator has been also derived in [24].

*Outline of paper.* In Section 2.1, we state the relevant definitions and notation of fractional order Sobolev spaces. Furthermore, in Section 2.2 we give a summary of the main results but reserve the proofs for subsequent sections. In Section 3, we introduce an auxiliary and a regularized version of the original problem and prove some local existence results for them. Finally, the local existence result for the original problem and then a finite time blow-up result are proved in Section 4.

## 2. OUTLINE OF RESULTS

**2.1. Fractional order Sobolev spaces.** In this subsection, we recall some well-known results on fractional order Sobolev spaces. To this end let  $\Omega \subset \mathbb{R}^N$  be an arbitrary open set with boundary  $\partial\Omega$ . For  $p \in [1, \infty)$  and  $s \in (0, 1)$ , we denote by

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy < \infty \right\}$$

the fractional order Sobolev space endowed with the norm

$$\|u\|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} |u|^p dx + \frac{C_{N,p,s}}{2} \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dx dy \right)^{\frac{1}{p}}.$$

In order to handle a non-smooth  $\Omega \subset \mathbb{R}^N$  in the case when  $\Omega$  is simply an open and bounded set, we let

$$W_0^{s,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)} \quad \text{and} \quad \widetilde{W}^{s,p}(\Omega) := \overline{W^{s,p}(\Omega) \cap C(\overline{\Omega})}^{W^{s,p}(\Omega)}.$$

By definition,  $W_0^{s,p}(\Omega)$  is the smallest closed subspace of  $\widetilde{W}^{s,p}(\Omega)$  containing the space  $\mathcal{D}(\Omega) := C_c^\infty(\Omega)$  (equipped with the topology that corresponds to convergence in the sense of test functions). If  $p \in (1, \infty)$ , then one may characterize the space  $W_0^{s,p}(\Omega)$  as follows (considering  $W_0^{s,p}(\Omega)$  as a subspace of  $\widetilde{W}^{s,p}(\Omega)$ )

$$W_0^{s,p}(\Omega) = \{u \in \widetilde{W}^{s,p}(\Omega) : \tilde{u} = 0 \text{ quasi-everywhere on } \partial\Omega\},$$

where  $\tilde{u}$  is the quasi-continuous version of  $u$  with respect to the capacity defined with the space  $\widetilde{W}^{s,p}(\Omega)$  (cf. [23, Theorem 4.5]). Finally we define the space

$$W_0^{s,p}(\overline{\Omega}) = \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. on } \mathbb{R}^N \setminus \Omega\}.$$

It is clear that  $W_0^{s,p}(\Omega)$  and  $W_0^{s,p}(\overline{\Omega})$  are both subspace of  $W^{s,p}(\Omega)$ , but there is no obvious inclusion between  $W_0^{s,p}(\Omega)$  and  $W_0^{s,p}(\overline{\Omega})$ . We notice that  $W_0^{s,p}(\overline{\Omega})$  contains the space of test functions  $\mathcal{D}(\Omega)$  but the latter space is not always dense in  $W_0^{s,p}(\overline{\Omega})$ . It has been proved in [14, Theorem 1.4.2.2] (see also [12] for some more general spaces) that if  $\Omega$  has a continuous boundary, then  $\mathcal{D}(\Omega)$  is dense in  $W_0^{s,p}(\overline{\Omega})$ . In addition if  $\Omega$  has a Lipschitz continuous boundary and  $s \neq 1/p$ , then  $W_0^{s,p}(\Omega) = W_0^{s,p}(\overline{\Omega})$  with equivalent norm.

Throughout the remainder of the paper, we make the convention that if we write  $u \in W_0^{s,p}(\overline{\Omega})$  we mean that  $u \in W^{s,p}(\mathbb{R}^N)$  and  $u = 0$  a.e. on  $\mathbb{R}^N \setminus \Omega$ . In that sense, a simple calculation shows that

$$\|u\|_{W_0^{s,p}(\overline{\Omega})} = \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \quad (2.1)$$

defines an equivalent norm on the space  $W_0^{s,p}(\overline{\Omega})$ . We shall always use this norm for the space  $W_0^{s,p}(\overline{\Omega})$  even when  $\Omega$  is simply an open bounded subset of  $\mathbb{R}^N$ . Let  $p^*$  be given by

$$p^* = \frac{Np}{N - sp} \text{ if } N > sp \text{ and } p^* \in [p, \infty) \text{ if } N = sp. \quad (2.2)$$

Then by [11, Section 7], there exists a constant  $C = C(N, p, s) > 0$  such that for every  $u \in W_0^{s,p}(\overline{\Omega})$ ,

$$\|u\|_{q,\Omega} \leq C \|u\|_{W_0^{s,p}(\overline{\Omega})}, \quad \forall q \in [p, p^*]. \quad (2.3)$$

Since  $\Omega$  is bounded, we have that (2.3) also holds for every  $q \in [1, p^*]$ . Moreover, the embedding  $W_0^{s,p}(\overline{\Omega}) \hookrightarrow L^q(\Omega)$  is compact for every  $q \in [1, p^*)$ . The following version of the Gagliardo-Nirenberg inequality for the space  $W_0^{s,p}(\overline{\Omega})$  in the non-smooth setting will be used. Let  $p \in (1, \infty)$ ,  $q, r \in [1, \infty]$  and  $0 \leq \alpha \leq 1$  satisfy

$$\frac{1}{q} = \frac{\alpha}{p^*} + \frac{1 - \alpha}{r} = \frac{N - sp}{Np} \alpha + \frac{1 - \alpha}{r}. \quad (2.4)$$

Then there exists a constant  $C > 0$  such that for every  $u \in W_0^{s,p}(\overline{\Omega})$ ,

$$\|u\|_{L^q(\Omega)} \leq \|u\|_{L^{p^*}(\Omega)}^\alpha \|u\|_{L^r(\Omega)}^{1-\alpha} \leq C \|u\|_{W_0^{s,p}(\overline{\Omega})}^\alpha \|u\|_{L^r(\Omega)}^{1-\alpha}. \quad (2.5)$$

If  $0 < s < 1$ ,  $p \in (1, \infty)$  and  $p' = p/(p-1)$ , the space  $W^{-s,p'}(\Omega)$  is defined as usual to be the dual of the reflexive Banach space  $W_0^{s,p}(\overline{\Omega})$ , that is,  $(W_0^{s,p}(\overline{\Omega}))^* = W^{-s,p'}(\Omega)$ . For more information on fractional order Sobolev spaces we refer the reader to [1, 11, 14, 15, 17, 23] and the references contained therein.

**2.2. Main results.** Let  $\Omega \subset \mathbb{R}^N$  be an arbitrary bounded open set. As usual for a Banach space  $X$ , we denote by  $C_w([a, b]; X)$  the set of all  $X$ -valued weakly continuous functions on the interval  $[a, b]$ . We also denote by  $\langle \cdot, \cdot \rangle_{X^*, X}$  the duality between  $X^*$  and  $X$ . First, we introduce the rigorous notion of solution to the system (1.1).

**Definition 2.1.** Let  $0 < s < 1$ ,  $2 \leq p, q, r < \infty$ ,  $p' = p/(p-1)$ ,  $r' = r/(r-1)$  and  $q' = q/(q-1)$ . Let  $u_0 \in L^r(\Omega)$  and

$$f \in W^{1,p'}((0, T); W^{-s,p'}(\Omega) + L^{r'}(\Omega)) \cap L^{1+\gamma}((0, T); L^r(\Omega))$$

for some  $\gamma \geq 0$  and  $T > 0$ . A function  $u$  is said to be a (strong) solution of (1.1) if

$$\begin{cases} u \in L^\infty((0, T); L^r(\Omega)) \cap L^p((0, T); W_0^{s,p}(\overline{\Omega})) \cap L^q((0, T); L^q(\Omega)), \\ \partial_t u \in L^{q'}((0, T); W^{-s,p'}(\Omega) + L^{r'}(\Omega)), \\ u(t) \in W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega), \text{ a.e. } t \in (0, T), \\ u \in W_{loc}^{1,2}((0, T); L^2(\Omega)), \end{cases} \quad (2.6)$$

and, a.e.  $t \in (0, T)$  for every  $v \in W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega) =: V$ , with  $r > \frac{N(q-p)}{sp}$ ,

$$\begin{aligned} & \langle \partial_t u(t), v \rangle_{V^*, V} \\ & + \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x, t) - u(y, t)|^{p-2} \frac{(u(x, t) - u(y, t))(v(x) - v(y))}{|x - y|^{N+sp}} dx dy \\ & = \langle |u(t)|^{q-2} u(t), v \rangle_{V^*, V} + \langle f(t), v \rangle_{V^*, V}, \end{aligned} \quad (2.7)$$

and  $u$  satisfies the initial condition

$$u(\cdot, t) \rightarrow u_0 \text{ strongly in } L^r(\Omega) \text{ as } t \rightarrow 0^+.$$

*Remark 2.2.* Notice that  $\langle |u(t)|^{q-2} u(t), v \rangle_{V^*, V}$  on the right-hand side of (2.7) is well-defined since for  $r > \frac{N(q-p)}{sp}$ ,  $V \subset L^q(\Omega)$  boundedly (see also (3.7) below).

The following is the first main result of the article.

**Theorem 2.3.** Let  $T > 0$  be fixed,  $0 < s < 1$  and  $p, q, r \in [2, \infty)$  be such that  $p < q$  and assume that

$$r > \frac{N(q-p)}{sp}.$$

Let  $u_0 \in L^r(\Omega)$  and assume

$$f \in W^{1,p'}((0, T); W^{-s,p'}(\Omega) + L^{r'}(\Omega)) \cap L^{1+\gamma}((0, T); L^r(\Omega))$$

for some  $\gamma \geq 0$ . Then the following assertions hold.

- (a) If  $\gamma > 0$ , then there exist a non-increasing function  $T_\star : [0, \infty) \times [0, \infty) \rightarrow [0, T]$  independent of  $T, u_0$  and  $f$  and

$$T_0 := T_\star \left( \|u_0\|_{L^r(\Omega)}, \int_0^T \|f(t)\|_{L^r(\Omega)}^{1+\gamma} dt \right),$$

such that (1.1) has at least one strong solution on  $(0, T_0)$ .

- (b) If  $\gamma = 0$ , then there exist a non-increasing function  $T_f : [0, \infty) \rightarrow (0, T]$  independent of  $T$  and  $u_0$ , and  $T_0 := T_f(\|u_0\|_{L^r(\Omega)})$ , such that (1.1) has at least one strong solution on  $(0, T_0)$ .

(c) *The strong solution has in addition the following regularity:*

$$\begin{cases} |u|^{\frac{r-2}{p}} u \in L^p((0, T_0); W_0^{s,p}(\overline{\Omega})), & (-\Delta)_p^s u \in L^{p'}((0, T_0); W^{-s,p'}(\Omega)), \\ t^{\frac{1}{p}} u \in C_w([0, T_0]; W_0^{s,p}(\overline{\Omega})), & \sqrt{t} \partial_t u \in L^2((0, T_0); L^2(\Omega)). \end{cases}$$

The proof of Theorem 2.3 relies on rewriting (1.1) as a first order Cauchy problem which is governed by the difference of two subdifferential operators in reflexive Banach spaces following the work of [2]. A family of approximate problems and refined energy estimates will be employed to construct solutions with initial data  $u_0 \in L^r(\Omega)$ . The primary new difficulty, due to the nonlocal character of the *fractional*  $p$ -Laplacian, is obtaining a new comparison lemma for various energy forms (see Lemma 3.9) and several other critical lemmas properly modified from [2] to handle our case. Solutions are first constructed for some auxiliary problems associated with (1.1).

The second main result deals with blow-up phenomena for the strong solutions of (1.1). To this end, we define the following energy functional

$$E(t) := \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x,t) - u(y,t)|^p}{|x-y|^{N+sp}} dx dy - \frac{1}{q} \int_{\Omega} |u(x,t)|^q dx \quad (2.8)$$

and notice that when  $f \equiv 0$ ,

$$\frac{d}{dt} E(t) = - \|\partial_t u(t)\|_{L^2(\Omega)}^2 \leq 0 \quad (2.9)$$

for as long as a *smooth* solution exists. In fact, every strong solution of Theorem 2.3 satisfies an energy inequality, as follows.

**Proposition 2.4.** *Let  $u$  be a solution in the sense of Theorem 2.3 and further assume that  $u_0 \in W_0^{s,p}(\overline{\Omega})$  and  $f \equiv 0$ . Then*

$$E(t) \leq E(0), \quad (2.10)$$

for almost all  $t \in (0, T_0)$ , for as long as a strong solution exists.

**Theorem 2.5.** *Let  $u$  be a strong solution of (1.1) in the sense of Theorem 2.3 and  $f \equiv 0$ . Let  $u_0 \in W_0^{s,p}(\overline{\Omega})$  such that  $E(0) < E_0$  and  $\|u_0\|_{W_0^{s,p}(\overline{\Omega})} > \alpha$  with*

$$\alpha = C_*^{-\frac{q}{q-p}}, \quad E_0 = \left( \frac{1}{p} - \frac{1}{q} \right) C_*^{-\frac{qp}{q-p}},$$

where  $C_* > 0$  is the best Sobolev constant in (2.3) and  $q \in (p, p^*]$ . Then the strong solution blows-up in a finite time  $t_* > 0$  with

$$t_* \leq \frac{\left(\frac{1}{2}\right)^{q-1} \|u_0\|_{L^2(\Omega)}^{q-2} |\Omega|^{\frac{q}{2}-1}}{\left(\frac{q}{2} - 1\right) \left(1 - \frac{\alpha^q}{\beta^q}\right) (q-p)}, \quad (2.11)$$

for some  $\beta > \alpha$ .

**Remark 2.6.** These results can be also extended to degenerate parabolic equations of the form

$$\partial_t u + L_{p,s,\Omega}(u) - g(u) = f(x, t), \quad (x, t) \in \Omega \times (0, T),$$

subject to the condition  $u = 0$  in  $\mathbb{R}^N \setminus \Omega$ , where

$$L_{p,s,\Omega}(u(x)) := \text{P.V.} \int_{\mathbb{R}^N} \mathbf{a}(u(x), u(y)) \left( \frac{u(x) - u(y)}{|x-y|^{N+ps}} \right) dy$$

with  $\mathfrak{a} \in C(\mathbb{R}^2, \mathbb{R}_+)$  satisfying the following condition

$$c_p |\tau_1 - \tau_2|^{p-2} \leq \mathfrak{a}(\tau_1, \tau_2) \leq c_0(1 + |\tau_1 - \tau_2|^{p-2}),$$

for all  $\tau_1, \tau_2 \in \mathbb{R}$ , for some  $c_0, c_p > 0$ . The function  $g$  is a maximal monotone graph in  $\mathbb{R}^2$  such that  $|g(s)| \leq c_g |s|^{q-1}$  as  $|s| \rightarrow \infty$ . We leave the details to the interested reader.

### 3. AUXILIARY AND REGULARIZED PROBLEMS

**3.1. Subdifferentials.** In this subsection we introduce some useful properties of subdifferentials of proper, convex and lower semi-continuous functionals on a Banach space.

**Definition 3.1.** Let  $X$  be a reflexive Banach space.

- (a) A mapping  $\varphi : X \rightarrow (-\infty, \infty]$  is called *proper* if its effective domain

$$D(\varphi) := \{x \in X : \varphi(x) < \infty\}$$

is not empty. For a proper mapping  $\varphi : X \rightarrow (-\infty, \infty]$ , we define the *convex conjugate*  $\varphi^*$  by

$$\varphi^* : X^* \rightarrow (-\infty, \infty], \quad \varphi^*(x^*) := \sup_{x \in X} x^*(x) - \varphi(x).$$

Note that  $\varphi^*$  is convex even if  $\varphi$  is not.

- (b) Given a mapping  $\varphi : X \rightarrow (-\infty, \infty]$  and  $x_0 \in X$ , a functional  $x^* \in X^*$  is called a *subgradient* of  $\varphi$  at  $x_0$  if for all  $x \in X$ , we have

$$x^*(x - x_0) \leq \varphi(x) - \varphi(x_0).$$

The set of all these subgradients is called the *subdifferential* of  $\varphi$  at  $x_0$  and is denoted by  $\partial_X \varphi(x_0)$ . The domain  $D(\partial_X \varphi)$  of the subdifferential  $\partial_X \varphi$  is given by

$$D(\partial_X \varphi) := \{x \in X : \partial_X \varphi(x) \neq \emptyset\}.$$

Obviously,  $D(\partial_X \varphi) \subset D(\varphi)$ .

It is well-known (see e.g. [6, 19]) that every subdifferential of a proper, convex and lower semi-continuous functional is maximal monotone. Moreover, if  $X = H$  is a Hilbert space then the subdifferential  $\partial_H \varphi$  can be written for  $u \in D(\varphi)$  as

$$\partial_H \varphi(u) = \{w \in H : \varphi(v) - \varphi(u) \geq (w, v - u)_H, \text{ for all } v \in D(\varphi)\},$$

where  $(\cdot, \cdot)_H$  denotes the inner product of  $H$ , and also  $\partial_H \varphi$  becomes a maximal monotone operator on  $H$ . For a proper, convex and lower semi-continuous functional  $\varphi$  on  $H$ , the *Moreau-Yosida approximation*  $\varphi_\lambda$  of  $\varphi$  is defined as follows:

$$\varphi_\lambda(u) := \inf_{v \in H} \left\{ \frac{1}{2\lambda} \|u - v\|_H^2 + \varphi(v) \right\}, \quad \text{for all } u \in H, \lambda > 0. \quad (3.1)$$

We recall that the *Yosida approximation* of a maximal monotone operator  $A$  on a Hilbert space  $H$  is defined as

$$A_\lambda := \frac{1}{\lambda} \left[ I - (I + \lambda A)^{-1} \right], \quad \lambda > 0. \quad (3.2)$$

The following result provides some useful properties of Moreau-Yosida and Yosida approximations. Its proof can be found in [6, Proposition 2.11, p.39].

**Proposition 3.2.** *Let  $\varphi$  be a proper, convex and lower semi-continuous functional on  $H$  and  $\varphi_\lambda$  be its Moreau-Yosida approximation. Then  $\varphi_\lambda$  is convex, Fréchet differentiable in  $H$ , and its Fréchet derivative  $\partial_H(\varphi_\lambda)$  coincides with the Yosida approximation  $(\partial_H\varphi)_\lambda$  of  $\partial_H\varphi$ . Moreover, the following properties hold:*

$$\begin{cases} \varphi_\lambda(u) = \frac{1}{2\lambda}\|u - J_\lambda^\varphi u\|_H^2 + \varphi(J_\lambda^\varphi u), & \text{for all } u \in H, \lambda > 0, \\ \varphi(J_\lambda^\varphi u) \leq \varphi_\lambda(u) \leq \varphi(u), & \text{for all } u \in H, \lambda > 0, \\ \varphi(J_\lambda^\varphi u) \uparrow \varphi(u) \text{ as } \lambda \rightarrow 0^+, & \text{for all } u \in H, \end{cases} \quad (3.3)$$

where  $J_\lambda^\varphi := (I + \lambda\partial_H\varphi)^{-1}$  is the resolvent operator of  $\partial_H\varphi$ .

The following type of chain rule for subdifferentials is taken from [2, Proposition 5].

**Proposition 3.3.** *Let  $X$  be a reflexive Banach space,  $T > 0$  be fixed and let  $\varphi : X \rightarrow (-\infty, \infty]$  be a proper, convex and lower semi-continuous functional. Let  $p \in (1, \infty)$  and let  $u \in W^{1,p}((0, T); X)$  be such that  $u(t) \in D(\partial_X\varphi)$  for a.e.  $t \in (0, T)$ . Suppose that there exists  $g \in L^{p'}((0, T); X^*)$  such that  $g(t) \in \partial_X\varphi(u(t))$  for a.e.  $t \in (0, T)$ . Then the function  $t \mapsto \varphi(u(t))$  is differentiable for a.e.  $t \in (0, T)$ . Moreover, for a.e.  $t \in (0, T)$ ,*

$$\frac{d}{dt}\varphi(u(t)) = \left\langle f, \frac{du}{dt}(t) \right\rangle_{X^*, X} \quad \text{for all } f \in \partial_X\varphi(u(t)). \quad (3.4)$$

Next, let  $\theta$  be a maximal monotone graph in  $\mathbb{R}^2$ . In the following result, for a given  $u \in L^2(\Omega)$  we discuss the representation of  $\theta(u(\cdot))$  as the subdifferential  $\partial_{L^2(\Omega)}\Theta(u)$  for some proper, convex and lower semi-continuous functional  $\Theta$  on  $L^2(\Omega)$ .

**Proposition 3.4.** *Let  $\Omega \subset \mathbb{R}^N$  be an open and bounded set and let  $\theta : \mathbb{R} \rightarrow (-\infty, \infty]$  be a proper, convex and lower semi-continuous functional. Define the functional  $\Theta : L^2(\Omega) \rightarrow (-\infty, \infty]$  with effective domain  $D(\Theta) = \{u \in L^2(\Omega) : \theta(u(\cdot)) \in L^1(\Omega)\}$  and given by*

$$\Theta(u) := \begin{cases} \int_\Omega \theta(u(x))dx & \text{if } u \in D(\Theta), \\ +\infty & \text{otherwise.} \end{cases}$$

Let  $J_\lambda^\Theta$  and  $j_\lambda^\theta$  ( $\lambda > 0$ ) denote the resolvent operators of the subdifferentials  $\partial_{L^2(\Omega)}\Theta$  and  $\partial_{\mathbb{R}}\theta$ , respectively. Then the following properties hold.

- (a) *The functional  $\Theta$  is proper, convex and lower semi-continuous on  $L^2(\Omega)$ .*
- (b) *For all  $f, u \in L^2(\Omega)$ , we have that  $f \in \partial_{L^2(\Omega)}\Theta(u)$  if and only if  $f(x) \in \partial_{\mathbb{R}}\theta(u(x))$  for a.e.  $x \in \Omega$ .*
- (c) *For all  $u \in L^2(\Omega)$ ,  $J_\lambda^\Theta u(x) = j_\lambda^\theta u(x)$  for a.e.  $x \in \Omega$  and for all  $\lambda > 0$ .*
- (d) *For every  $m \in [1, \infty]$ , if  $u, v \in L^m(\Omega) \cap L^2(\Omega)$ , then  $J_\lambda^\Theta u, \partial_{L^2(\Omega)}\Theta_\lambda(u) \in L^m(\Omega) \cap L^2(\Omega)$  for all  $\lambda > 0$  and*

$$\begin{aligned} \|J_\lambda^\Theta u - J_\lambda^\Theta v\|_{L^m(\Omega)} &\leq \|u - v\|_{L^m(\Omega)}, \\ \|\partial_{L^2(\Omega)}\Theta_\lambda(u) - \partial_{L^2(\Omega)}\Theta_\lambda(v)\|_{L^m(\Omega)} &\leq \frac{2}{\lambda}\|u - v\|_{L^m(\Omega)}. \end{aligned}$$



- (e) If  $\partial_{\mathbb{R}}\theta(0) \ni 0$ , then for every  $p \in (1, \infty)$  and  $s \in (0, 1)$ , we have that  $J_{\lambda}^{\Theta}0 = 0$ ,  $J_{\lambda}^{\Theta}u \in W_0^{s,p}(\overline{\Omega}) \cap L^2(\Omega)$  for all  $u \in W_0^{s,p}(\overline{\Omega}) \cap L^2(\Omega)$  and for all  $\lambda > 0$ . Moreover,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|J_{\lambda}^{\Theta}u(x) - J_{\lambda}^{\Theta}u(y)|^p}{|x - y|^{N+sp}} dx dy \leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy. \quad (3.5)$$

*Proof.* The proof of parts (a), (b) (c) and (d) is contained in [2, Proposition 6] (see also [19, Proposition 8.1] for parts (a) and (b) and also [6, Proposition 2.16, p.47]).

Next, let  $\lambda > 0$ ,  $p \in (1, \infty)$ ,  $s \in (0, 1)$  and  $u \in W^{s,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ . It follows from part (d) that  $J_{\lambda}^{\Theta}u \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ . Since

$$|j_{\lambda}^{\Theta}(u(x)) - j_{\lambda}^{\Theta}(u(y))| \leq |u(x) - u(y)| \text{ for a.e. } x, y \in \mathbb{R}^N,$$

then we obtain (3.5) by using the assertion (c).

It remains to show the assertion (e). First, let  $\lambda > 0$ ,  $p \in (1, \infty)$ ,  $s \in (0, 1)$  and  $u \in W^{s,p}(\mathbb{R}^N) \cap L^2(\Omega)$ . It follows from part (d) that  $J_{\lambda}^{\Theta}u \in L^p(\Omega) \cap L^2(\Omega)$ . Since

$$|j_{\lambda}^{\Theta}(u(x)) - j_{\lambda}^{\Theta}(u(y))| \leq |u(x) - u(y)| \text{ for a.e. } x, y \in \mathbb{R}^N, \quad (3.6)$$

then we obtain (3.5) by using the assertion (c). Next, assume that  $\partial_{\mathbb{R}}\theta(0) \ni 0$ . Then it is clear that  $j_{\lambda}^{\Theta}0 = 0$  and hence,  $J_{\lambda}^{\Theta}0 = 0$  and  $|J_{\lambda}^{\Theta}u(x)| \leq |u(x)|$  for a.e.  $x \in \Omega$ . Let  $u \in W_0^{s,p}(\overline{\Omega}) \cap L^2(\Omega) \subset W^{s,p}(\mathbb{R}^N) \cap L^2(\Omega)$  and  $\lambda > 0$ . Since  $J_{\lambda}^{\Theta}u \in L^p(\Omega) \cap L^2(\Omega)$  (see above), it follows from (3.6) and part (c) that  $J_{\lambda}^{\Theta}u \in W^{s,p}(\mathbb{R}^N) \cap L^2(\Omega)$ . Since  $|J_{\lambda}^{\Theta}u(x)| \leq |u(x)|$  for a.e.  $x \in \mathbb{R}^N$  we also have that  $J_{\lambda}^{\Theta}u = 0$  on  $\mathbb{R}^N \setminus \Omega$ . Therefore  $J_{\lambda}^{\Theta}u \in W_0^{s,p}(\overline{\Omega}) \cap L^2(\Omega)$  and we have shown part (e). The proof of the proposition is finished.  $\square$

**3.2. The auxiliary problems.** We first write the system (1.1) as a first order Cauchy problem. To this end recall that  $0 < s < 1$ ,  $p, r \in [2, \infty)$  and denote  $V := W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$  as the Banach space equipped with the norm

$$\|u\|_V := \left( \|u\|_{L^r(\Omega)}^2 + \|u\|_{W_0^{s,p}(\overline{\Omega})}^2 \right)^{\frac{1}{2}}.$$

where the second norm is given by (2.1). Let  $V^*$  denote the dual of the reflexive Banach space  $V$ . Then

$$V^* = W^{-s,p'}(\Omega) + L^{r'}(\Omega) := \{u = u_1 + u_2; u_1 \in W^{-s,p'}(\Omega), u_2 \in L^{r'}(\Omega)\},$$

where  $p' = p/(p-1)$  and  $r' = r/(r-1)$ . For every  $r \geq 2$ , we have the continuous injections  $V \hookrightarrow L^2(\Omega) \hookrightarrow V^*$ .

Next, for  $p, q, r \in [2, \infty)$  satisfying  $p < q$  and

$$r > \frac{N(q-p)}{sp} \iff q < \frac{N+sr}{N}p, \quad (3.7)$$

we have that  $V$  is continuously embedded into  $L^q(\Omega)$ . Indeed, if  $q \leq r$ , then  $V$  is trivially continuously embedded into  $L^q(\Omega)$ . If  $r < q$ , then it follows from (3.7) that  $q < p^*$ . Since  $W_0^{s,p}(\overline{\Omega}) \hookrightarrow L^{p^*}(\Omega)$ , we also have  $V \hookrightarrow L^q(\Omega)$ .

Let us now define the functionals  $\Phi, \psi : V \rightarrow [0, \infty)$  by

$$\Phi(u) := \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy, \quad \psi(u) := \frac{1}{q} \int_{\Omega} |u|^q dx,$$

for all  $u \in V$ . It is easy to see that  $\Phi, \psi \in C^1(V, \mathbb{R})$ . We state the following basic proposition whose proof is postponed until the Appendix.

**Proposition 3.5.** *Let  $\partial_V \Phi$  and  $\partial_V \psi$  denote the single valued subgradients of  $\Phi$  and  $\psi$ , respectively. Then  $\partial_V \Phi$  is an operator from  $V$  to  $V^*$  and can be expressed as*

$$D(\partial_V \Phi) = V \quad \text{and} \quad \partial_V \Phi(u) = (-\Delta)_p^s u, \quad \text{for all } u \in V.$$

*More precisely,  $\partial_V \Phi$  is a realization in  $V^*$  of the fractional  $p$ -Laplace operator  $(-\Delta)_p^s$  with the Dirichlet boundary condition  $u = 0$  on  $\mathbb{R}^N \setminus \Omega$ . Finally, under the assumption (3.7), we also have that  $\partial_V \psi$  is an operator from  $V$  to  $V^*$  with*

$$D(\partial_V \psi) = V \quad \text{and} \quad \partial_V \psi(u) = |u|^{q-2} u, \quad \text{for all } u \in V.$$

By virtue of Proposition 3.5, the system (1.1) can be rewritten as the following abstract Cauchy problem

$$\begin{cases} \frac{du}{dt}(t) + \partial_V \Phi(u(t)) - \partial_V \psi(u(t)) = f(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases} \quad (3.8)$$

Next, we also define the functional  $\phi : L^2(\Omega) \rightarrow [0, \infty]$  by

$$\phi(u) := \begin{cases} \frac{1}{r} \int_{\Omega} |u|^r dx & \text{if } u \in L^r(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

We note that the energy functional  $\Phi(u) - \psi(u)$  is not bounded from below on  $W_0^{s,p}(\overline{\Omega}) \cap L^q(\Omega)$  but the sum  $\Phi(u) - \psi(u) + I_{\mathcal{X}}$ , where  $I_{\mathcal{X}}$  denotes the characteristic function over some ball  $\mathcal{X}$  in  $L^r(\Omega)$  turns out to be coercive provided that  $r$  satisfies (3.7). In this respect, we can establish the following crucial result.

**Lemma 3.6.** *Let  $0 < s < 1$  and let  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Then there exist a constant  $\varepsilon \in (0, 1]$  and an increasing differentiable function  $\mathcal{F} : [0, \infty) \rightarrow [0, \infty)$  such that for every  $u \in D(\Phi) \cap D(\psi) = W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega) = V$ ,*

$$\psi(u) \leq \mathcal{F}(\phi(u))[\Phi(u) + 1]^{1-\varepsilon}. \quad (3.9)$$

*Proof.* Let  $u \in W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$ . If  $q \leq r$ , since  $\Omega$  is bounded then by the classical Hölder inequality, we have that there exists a constant  $C > 0$  such that

$$\frac{1}{q} \int_{\Omega} |u|^q dx = \psi(u) \leq C \left( \frac{1}{r} \int_{\Omega} |u|^r dx \right)^{\frac{q}{r}} = C[\phi(u)]^{\frac{q}{r}}.$$

Hence, we have that (3.9) holds with  $\varepsilon = 1$  and  $\mathcal{F}(t) = Ct^{q/r}$ .

If  $r < q$ , then  $q < p^*$  (see (2.2)). Hence, using the Gagliardo-Nirenberg's inequality (2.5), one can find a constant  $C > 0$  such that

$$\|u\|_{L^q(\Omega)} \leq C \left( \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{\alpha}{p}} \|u\|_{L^r(\Omega)}^{1-\alpha}, \quad (3.10)$$

where  $\alpha \in (0, 1)$  is given by

$$\frac{1}{q} = \frac{\alpha}{p^*} + \frac{1-\alpha}{r} = \frac{N-sp}{Np} \alpha + \frac{1-\alpha}{r}. \quad (3.11)$$

We notice that (3.11) and (3.7) imply that

$$0 < \alpha q = (q-r) \left( 1 - \frac{N-sp}{Np} r \right) < \frac{\left( \frac{N+sr}{N} \right) p - r}{1 - \left( \frac{N-sp}{Np} \right) r} = p. \quad (3.12)$$

It follows from (3.10) that

$$\begin{aligned}\psi(u) &\leq C \left( \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{\alpha q}{p}} \|u\|_{L^r(\Omega)}^{(1-\alpha)q} \\ &= C [\Phi(u)]^{\frac{\alpha q}{p}} [\phi(u)]^{\frac{(1-\alpha)q}{r}}.\end{aligned}$$

Note that  $0 < \alpha q/p < 1$  by (3.12). Thus we have shown (3.9) with the constant  $\varepsilon = 1 - \alpha q/p$  and  $\mathcal{F}(t) = Ct^{\frac{(1-\alpha)q}{r}}$ . The proof of lemma is finished.  $\square$

Next, let  $T > 0$  be fixed,  $u_0 \in D(\Phi) = V = W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$  and  $f \in C^1([0, T]; V)$ . We shall introduce an auxiliary problem associated with the abstract Cauchy problem (3.8). To do this, we let  $\sigma := \phi(u_0) + 1$  and set

$$V_\sigma = \{v \in V : \phi(v) \leq \sigma \iff \|v\|_{L^r(\Omega)}^r \leq r\sigma\}.$$

We define the proper, convex, lower semi-continuous functional  $\Phi^\sigma : V \rightarrow [0, \infty]$  by

$$\Phi^\sigma(u) := \begin{cases} \Phi(u) & \text{if } u \in V_\sigma, \\ +\infty & \text{otherwise.} \end{cases}$$

Clearly,  $D(\Phi^\sigma) = V_\sigma \subset V = D(\Phi)$  and  $D(\partial_V \Phi^\sigma) = V_\sigma \subset V = D(\partial_V \Phi)$ . It follows from [7, Theorem 2.2] that for all  $u \in D(\partial_V \Phi^\sigma)$ ,

$$\partial_V \Phi^\sigma(u) = \partial_V \Phi(u) + \partial_V \chi_{V_\sigma}(u) \quad (3.13)$$

where  $\chi_{V_\sigma}$  denotes the indicator function of the convex set  $V_\sigma$  defined by

$$\chi_{V_\sigma}(u) = \begin{cases} 0 & u \in V_\sigma \\ \infty & u \notin V_\sigma. \end{cases}$$

We notice that by [6, Example 2.8.2], the subdifferential  $\partial_V \chi_{V_\sigma}$  of the functional  $\chi_{V_\sigma}$  is given by

$$\partial_V \chi_{V_\sigma}(u) = \begin{cases} \emptyset & \text{if } u \notin V_\sigma, \\ \{0\} & \text{if } u \in \text{Int}(V_\sigma), \\ \text{the normal exterior cone to } V_\sigma & \text{if } u \in \text{boundary}(V_\sigma). \end{cases} \quad (3.14)$$

Corresponding to problem (3.8) we consider the following modified problem

$$\begin{cases} \frac{du}{dt}(t) + \partial_V \Phi^\sigma(u(t)) - \partial_V \psi(u(t)) \ni f(t) & \text{in } V^*, \quad 0 < t < T, \\ u(0) = u_0. \end{cases} \quad (3.15)$$

We observe that a solution of problem (3.15) on  $(0, T)$  is also a solution of (3.8) on  $(0, T)$  provided that one has in addition  $\phi(u(t)) < \sigma$ . Indeed, in that case  $\partial_V \chi_{V_\sigma}(u(t)) = \{0\}$  by (3.14), and by (3.13), this implies  $\partial_V \Phi^\sigma(u(t)) = \partial_V \Phi(u(t))$  a.e.  $t \in (0, T)$ . Thus, in order to establish the existence of a solution to problem (3.8) it suffices to construct a sufficiently regular solution to the Cauchy problem (3.15) and to derive additional a priori estimates on this solution. To this end, we first define the extensions  $\overline{\Phi}^\sigma, \overline{\psi}$  of  $\Phi^\sigma$  and  $\psi$ , respectively, to the Hilbert space  $H := L^2(\Omega)$  by

$$\overline{\Phi}^\sigma(u) := \begin{cases} \Phi^\sigma(u) & \text{if } u \in V, \\ +\infty & \text{otherwise.} \end{cases}$$

and

$$\bar{\psi}(u) := \begin{cases} \frac{1}{q} \int_{\Omega} |u|^q dx & \text{if } u \in L^q(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then,  $\bar{\Phi}^\sigma$  and  $\bar{\psi}$  are proper, convex and lower semi-continuous on  $H = L^2(\Omega)$ . Let  $\partial_H \bar{\Phi}^\sigma$  and  $\partial_H \bar{\psi}$  denote the subdifferentials of  $\bar{\Phi}^\sigma$  and  $\bar{\psi}$ , respectively. Then, it readily follows

$$\begin{cases} D(\bar{\Phi}^\sigma) = D(\Phi^\sigma), & D(\partial_H \bar{\Phi}^\sigma) \subset D(\partial_V \Phi^\sigma), \\ \partial_H \bar{\Phi}^\sigma(u) \subset \partial_V \Phi^\sigma(u), & \text{for all } u \in D(\partial_H \bar{\Phi}^\sigma), \end{cases} \quad (3.16)$$

and

$$\begin{cases} \bar{\psi}(u) = \psi(u) \quad \forall u \in V, & D(\partial_H \bar{\psi}) \cap V \subset D(\partial_V \psi), \\ \partial_H \bar{\psi}(u) \subset \partial_V \psi(u) & \text{for all } u \in D(\partial_H \bar{\psi}) \cap V. \end{cases} \quad (3.17)$$

Now consider  $\bar{\psi}_\lambda$  as the Moreau-Yosida approximation (see (3.1)) of  $\bar{\psi}$ , for  $\lambda > 0$ . Associated with problem (3.15), we introduce the following regularized problem in  $H = L^2(\Omega)$ ,

$$\begin{cases} \frac{du_\lambda}{dt}(t) + \partial_H \bar{\Phi}^\sigma(u_\lambda(t)) - \partial_H \bar{\psi}_\lambda(u_\lambda(t)) \ni f(t) & \text{in } H = L^2(\Omega), \quad 0 < t < T, \\ u_\lambda(0) = u_0. \end{cases} \quad (3.18)$$

Regarding the functionals defined above, we mention the following facts.

*Remark 3.7.* (a) It follows from Lemma 3.6 that for every  $u \in D(\Phi^\sigma) = V_\sigma$ ,

$$\psi(u) \leq \mathcal{F}[\phi(u)] [\Phi(u) + 1]^{1-\varepsilon} \leq \frac{1}{2} \Phi(u) + \mathcal{F}(\sigma). \quad (3.19)$$

(b) There exists a constant  $C > 0$  such that for every  $u \in D(\Phi^\sigma) = V_\sigma$ ,

$$\begin{aligned} \|u\|_V^p &= \|u\|_{W_0^{s,p}(\bar{\Omega}) \cap L^r(\Omega)}^p = \left( \|u\|_{L^r(\Omega)}^2 + \|u\|_{W_0^{s,p}(\bar{\Omega})}^2 \right)^{\frac{p}{2}} \\ &\leq C \left( \|u\|_{L^r(\Omega)}^p + \|u\|_{W_0^{s,p}(\bar{\Omega})}^p \right) \leq C \left( \Phi^\sigma(u) + \sigma^{\frac{p}{r}} \right). \end{aligned} \quad (3.20)$$

(c) The subdifferential  $\partial_V \psi : V \rightarrow V^*$  is a compact operator. Indeed, let  $C \geq 0$  and let  $u_n$  be a sequence in  $V$  such that  $\|u_n\|_V \leq C$ . Then, after a subsequence if necessary,  $u_n$  converges weakly to some  $u$  in the reflexive Banach space  $V$ . Since the embedding  $V \hookrightarrow L^q(\Omega)$  is compact, passing to a subsequence if necessary, we may assume that

$$u_n \rightarrow u \text{ strongly in } L^q(\Omega).$$

Since  $\partial_V \psi(u_n) = |u_n|^{q-2} u_n$  we have that

$$\partial_V \psi(u_n) \rightarrow \partial_V \psi(u) \text{ strongly in } L^{q'}(\Omega).$$

Since  $L^{q'}(\Omega) \hookrightarrow V^*$ , it follows that

$$\partial_V \psi(u_n) \rightarrow \partial_V \psi(u) \text{ strongly in } V^*.$$

Hence,  $\partial_V \psi : V \rightarrow V^*$  is a compact operator.

(d) Finally, let  $J_\lambda^{\bar{\psi}}$  ( $\lambda > 0$ ) be the resolvent operator of  $\partial_H \bar{\psi}$ . By Proposition 3.4, parts (e) and (f), we readily have

$$\phi(J_\lambda^{\bar{\psi}} u) \leq \phi(u) \leq \sigma, \quad \Phi(J_\lambda^{\bar{\psi}} u) \leq \Phi(u), \text{ for all } u \in D(\Phi^\sigma). \quad (3.21)$$

Moreover,

$$\Phi^\sigma(J_\lambda^{\bar{\psi}} u) \leq \Phi^\sigma(u), \text{ for all } u \in D(\Phi^\sigma). \quad (3.22)$$

We conclude this subsection with the following lemma.

**Lemma 3.8.** *Recall  $0 < s < 1$  and let  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Let  $u \in L^r(\Omega)$  be such that  $v := |u|^{\frac{r-2}{p}} u \in W_0^{s,p}(\bar{\Omega})$ . Let  $\bar{\psi}_\lambda$  and  $\phi_\lambda$  ( $\lambda > 0$ ) be the Morreau-Yosida approximation of  $\bar{\psi}$  and  $\phi$ , respectively. Then there exist a constant  $\varepsilon \in (0, 1]$  and an increasing differentiable function  $\mathcal{F} : [0, \infty) \rightarrow [0, \infty)$  such that*

$$\begin{aligned} & \int_{\Omega} \partial_H \bar{\psi}_\lambda(u) \partial_H \phi_\lambda(u) dx \\ & \leq \mathcal{F}(\phi(u)) \left[ 1 + \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right]^{1-\varepsilon}. \end{aligned} \quad (3.23)$$

*Proof.* Let  $\lambda > 0$  and let  $u \in L^r(\Omega)$  be such that  $v := |u|^{\frac{r-2}{p}} u \in W_0^{s,p}(\bar{\Omega})$ . Since  $|J_\lambda^\psi u(x)| \leq |u(x)|$  and  $|J_\lambda^\phi u(x)| \leq |u(x)|$  for a.e.  $x \in \Omega$ , we have that

$$\begin{aligned} & \int_{\Omega} \partial_H \bar{\psi}_\lambda(u) \partial_H \phi_\lambda(u) dx \\ & = \int_{\Omega} |J_\lambda^\psi u(x)|^{q-2} J_\lambda^\psi u(x) |J_\lambda^\phi u(x)|^{r-2} J_\lambda^\phi u(x) dx \leq \int_{\Omega} |u(x)|^{q+r-2} dx. \end{aligned}$$

In light of (3.7) we easily see that

$$q + r - 2 < \left( \frac{N + rs}{N} \right) p + r - 2 = \left( 1 + \frac{sp}{N} \right) r + p - 2. \quad (3.24)$$

If  $sp < N$ , then

$$\begin{aligned} \left( \frac{r + p - 2}{p} \right) p^* &= \left( \frac{N}{N - sp} \right) r + \left( \frac{N}{N - sp} \right) (p - 2) \\ &= \left( 1 + \frac{sp}{N - sp} \right) r + \left( \frac{N}{N - sp} \right) (p - 2). \end{aligned} \quad (3.25)$$

It follows from (3.24) and (3.25) that

$$\rho := p \left( \frac{q + r - 2}{r + p - 2} \right) < p^*.$$

Next, let  $v := |u|^{\frac{r-2}{p}} u$ . Then

$$|u|^{q+r-2} = |v|^\rho \quad \text{and} \quad |u|^r = |v|^{\frac{pr}{r+p-2}}.$$

Assume that  $v \in W_0^{s,p}(\bar{\Omega})$ . Since  $1 < \frac{pr}{r+p-2} < \rho < p^*$  and

$$\|v\|_{W_0^{s,p}(\bar{\Omega})} \leq \left[ \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)} + \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \right],$$

then using the Gagliardo-Nirenberg inequality (2.5), we have that there exists a constant  $C > 0$  such that

$$\begin{aligned} \|v\|_{L^p(\Omega)} &\leq C \|v\|_{W_0^{s,p}(\overline{\Omega})}^\alpha \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)}^{1-\alpha} \\ &\leq C \left[ \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)} + \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \right]^\alpha \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)}^{1-\alpha} \\ &\leq C \left[ \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \right]^\alpha \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)}^{1-\alpha} + C \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)} \end{aligned} \quad (3.26)$$

with  $0 \leq \alpha \leq 1$  satisfying

$$\frac{1}{\rho} = \frac{\alpha}{p^*} + \frac{(1-\alpha)(r+p-2)}{rp}.$$

A simple calculation gives

$$0 < \alpha = \frac{\left(\frac{r+p-2}{pr}\right) - \left(\frac{r+p-2}{p(q+r-2)}\right)}{\left(\frac{r+p-2}{pr}\right) - \left(\frac{N-sp}{Np}\right)} < 1.$$

It follows from (3.26) that,

$$\begin{aligned} \int_{\Omega} |u|^{q+r-2} dx &= \int_{\Omega} |v|^\rho dx \\ &\leq C \left[ \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \right]^{\alpha\rho} \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)}^{(1-\alpha)\rho} + C \|v\|_{L^{\frac{rp}{r+p-2}}(\Omega)}^\rho \\ &\leq C \left[ \left( \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}} \right]^{\alpha\rho} \|v\|_{L^r(\Omega)}^{(1-\alpha)(q+r-2)} + C \|v\|_{L^r(\Omega)}^{q+r-2}. \end{aligned} \quad (3.27)$$

Letting

$$\mathcal{F}(t) := \sup \left\{ t^{\frac{(1-\alpha)(q+r-2)}{r}}, t^{\frac{q+r-2}{r}} \right\}, \quad t \geq 0$$

we have that  $\mathcal{F} : [0, \infty) \rightarrow [0, \infty)$  is increasing and differentiable. Thus, (3.23) follows from (3.27) together with the simple estimate

$$1 - \varepsilon = \frac{\alpha\rho}{p} = \frac{1}{p} \frac{q-2}{\left(\frac{r+p-2}{p}\right) - \left(\frac{N-sp}{Np}\right)} < \frac{1}{p} \frac{\left(\frac{N+rs}{N}\right)p-2}{\left(\frac{r+p-2}{p}\right) - \left(\frac{N-sp}{Np}\right)r} = 1.$$

The proof of the lemma is finished.  $\square$

**3.3. Solutions to the auxiliary problems.** In this subsection, we investigate the existence and regularity of solutions to problems (3.15), (3.18) for regular initial datum  $u_0 \in D(\Phi) = V$  and  $f \in C^1([0, T]; V)$ . Before we turn our attention directly to the Cauchy problem (3.18), we require the following two crucial lemmas. The first result is essential and is of independent interest. The second one establishes a kind of coercivity estimate. Their proofs are postponed until the Appendix.

**Lemma 3.9.** *Let  $p \in (1, \infty)$ ,  $r \in [2, \infty)$  and let  $\mathcal{E}$  be the energy given by*

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} (u(x) - u(y))(v(x) - v(y)) K(x, y) dx dy,$$

for some positive kernel  $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ . Then

$$C_{r,p} \mathcal{E}(|u|^{\frac{r-2}{p}} u, |u|^{\frac{r-2}{p}} u) \leq \mathcal{E}(u, |u|^{r-2} u), \quad (3.28)$$

for all functions  $u$  for which the terms in (3.28) make sense, and where

$$C_{r,p} := (r-1) \left( \frac{p}{p+r-2} \right)^p.$$

**Lemma 3.10.** *Let  $0 < s < 1$ ,  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Let  $u \in D(\partial_V \Phi)$  and let  $J_\mu^\phi := (I + \mu \partial_H \phi)^{-1}$ ,  $\mu > 0$ . Then*

$$J_\mu^\phi u \in D(\partial_V \Phi), \quad \partial_H \phi_\mu(u) \in V, \quad v_\mu := |J_\mu^\phi u|^{\frac{r-2}{p}} J_\mu^\phi u \in W_0^{s,p}(\overline{\Omega}).$$

*In particular, if  $u \in D(\partial_V \Phi^\sigma)$ , then there exists a positive constant  $\beta$  independent of  $\mu$  such that for all  $g \in \partial_V \Phi^\sigma(u)$ ,*

$$\frac{\beta C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\mu(x) - v_\mu(y)|^p}{|x - y|^{N+sp}} dx dy \leq \langle g, \partial_H \phi_\mu(u) \rangle_{V^*, V}. \quad (3.29)$$

We have the following result of existence of solutions to the abstract Cauchy problem (3.18).

**Proposition 3.11.** *Let  $0 < s < 1$ ,  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Let  $T > 0$  be fixed,  $u_0 \in D(\Phi)$ ,  $\lambda > 0$  and  $f \in C^1([0, T]; V)$ . Then there exists a unique function  $u_\lambda \in C_w([0, T]; V) \cap W^{1,2}((0, T); L^2(\Omega))$  which is a strong solution of (3.18) on  $(0, T)$ . Moreover,*

$$\sup_{t \in [0, T]} \phi(u_\lambda(t)) \leq \sigma \quad \text{and} \quad v_\lambda := |u_\lambda|^{\frac{r-2}{p}} u_\lambda \in L^p((0, T); W_0^{s,p}(\overline{\Omega})). \quad (3.30)$$

*In addition, the function  $t \mapsto \overline{\Phi}^\sigma(u_\lambda(t))$  is absolutely continuous on  $[0, T]$ .*

*Proof.* First, we notice that by Proposition 3.2,  $\partial_H \overline{\psi}_\lambda$  coincides with the Yosida approximation  $(\partial_H \overline{\psi})_\lambda$  of the maximal monotone operator  $\partial_H \overline{\psi}$  (see (3.2)). Hence, by Proposition 3.4  $\partial_H \overline{\psi}_\lambda$  is Lipschitz continuous in  $L^r(\Omega)$  as well as in  $L^2(\Omega)$ . Since  $\overline{\Phi}^\sigma$  is proper, convex and lower semi-continuous on  $H = L^2(\Omega)$  and the mapping  $t \mapsto f(t)$  belongs to  $L^2((0, T); L^2(\Omega))$ , we can exploit [6, Proposition 3.12 and Theorem 3.6] to infer that for every  $u_0 \in \overline{D(\overline{\Phi}^\sigma)} = L^2(\Omega)$ , the Cauchy problem (3.18) has a unique strong solution  $u_\lambda$ . Moreover, it holds

$$\sqrt{t} \frac{du_\lambda}{dt}(t) \in L^2((0, T); L^2(\Omega)).$$

In particular, if  $u_0 \in D(\overline{\Phi}^\sigma)$  we have

$$u_\lambda \in C_w([0, T]; V) \cap W^{1,2}((0, T); L^2(\Omega)) \quad \text{and} \quad u_\lambda(t) \in V, \quad \phi(u_\lambda(t)) \leq \sigma,$$

for all  $t \in [0, T]$ . Hence, the function  $t \mapsto \Phi^\sigma(u_\lambda(t))$  is absolutely continuous on  $[0, T]$  and the first statement of (3.30) also follows. It remains to show the second part of (3.30). To this end, multiplying (3.18) by  $\partial_H \phi_\mu(u_\lambda(t))$ ,  $\mu > 0$ , then employing the chain rule formula (3.4) (see Proposition 3.3), we obtain

$$\begin{aligned} \frac{d}{dt} \phi_\mu(u_\lambda(t)) + \int_{\Omega} \partial_H \phi_\mu(u_\lambda(t)) \left[ f(t) - \frac{du_\lambda}{dt}(t) + \partial_H \overline{\psi}_\lambda(u_\lambda(t)) \right] dx \\ = \int_{\Omega} \partial_H \overline{\psi}_\lambda(u_\lambda(t)) \partial_H \phi_\mu(u_\lambda(t)) dx + \int_{\Omega} f(t) \partial_H \phi_\mu(u_\lambda(t)) dx. \end{aligned} \quad (3.31)$$

Let  $v_{\lambda,\mu}(t) := |J_\mu^\phi u_\lambda(t)|^{\frac{r-2}{p}} J_\mu^\phi u_\lambda(t)$ . Note that  $v_{\lambda,\mu}(t) \in W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega) = V$ . Recall that by Lemma 3.10 and virtue of (3.16) it holds

$$\begin{aligned} & \beta \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,t) - v_{\lambda,\mu}(y,t)|^p}{|x-y|^{N+sp}} dx dy \\ & \leq \langle \partial_H \overline{\Phi}^\sigma(u_\lambda(t)), \partial_H \phi_\mu(u_\lambda(t)) \rangle_{V^*, V}. \end{aligned} \quad (3.32)$$

By Proposition 3.4,  $\partial_H \overline{\psi}_\lambda$  is Lipschitz continuous from  $L^r(\Omega)$  to  $L^r(\Omega)$ . Hence, Hölder's inequality together with Proposition 3.2 yields the estimate

$$\begin{aligned} \int_{\Omega} \partial_H \overline{\psi}_\lambda(u_\lambda(t)) \partial_H \phi_\mu(u_\lambda(t)) dx & \leq \|\partial_H \overline{\psi}_\lambda(u_\lambda(t))\|_{L^r(\Omega)} \|\partial_H \phi_\mu(u_\lambda(t))\|_{L^{r'}(\Omega)} \\ & \leq C_\lambda \phi(u_\lambda(t)) \leq C_\lambda \sigma, \end{aligned} \quad (3.33)$$

for some constant  $C_\lambda > 0$  depending only on  $\lambda > 0$  but not on  $\mu > 0$ . Moreover, exploiting Hölder's inequality once again, one can find a constant  $C > 0$  such that

$$\int_{\Omega} f(t) \partial_H \phi_\mu(u_\lambda(t)) dx \leq C \sigma^{\frac{1}{r}} \|f(t)\|_{L^r(\Omega)}.$$

Combining (3.32) together with (3.33), then integrating (3.31) over  $(0, t)$ , and using Proposition 3.2 once more, we deduce

$$\begin{aligned} & \phi_\mu(u_\lambda(t)) + \beta \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,\tau) - v_{\lambda,\mu}(y,\tau)|^p}{|x-y|^{N+sp}} dx dy d\tau \\ & \leq \phi(u_0) + C_\lambda \sigma T + \beta \sigma^{\frac{1}{r}} \int_0^T \|f(\tau)\|_{L^r(\Omega)} d\tau, \end{aligned} \quad (3.34)$$

for all  $t \in [0, T]$ . Now passing to the limit as  $\mu \rightarrow 0^+$  in the foregoing uniform estimate, by virtue of Proposition 3.2, we obtain

$$J_\mu^\phi(u_\lambda) \rightarrow u_\lambda \quad \text{strongly in } C([0, T]; L^2(\Omega)).$$

Finally, since  $2(p+r-2)/p \leq r$ , it follows from (3.34) that as  $\mu \rightarrow 0^+$ ,

$$\begin{cases} J_\mu^\phi(u_\lambda) \rightarrow u_\lambda & \text{weakly-star in } L^\infty((0, T); L^r(\Omega)), \\ v_{\lambda,\mu} \rightarrow v_\lambda & \text{weakly-star in } L^\infty((0, T); L^2(\Omega)), \\ v_{\lambda,\mu} \rightarrow v_\lambda & \text{weakly in } L^p((0, T); W_0^{s,p}(\overline{\Omega})), \end{cases}$$

where  $v_\lambda = |u_\lambda|^{\frac{r-2}{p}} u_\lambda$ . The proof is finished.  $\square$

Having obtained a solution to the regularized problem (3.18), we can now pass to the limit as  $\lambda \rightarrow 0^+$  to deduce a solution to problem (3.15). We have the following.

**Proposition 3.12.** *Let  $0 < s < 1$ ,  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Let  $T > 0$  be fixed,  $u_0 \in D(\Phi)$  and  $f \in C^1([0, T]; V)$ . Then there exists a unique function  $u \in C_w([0, T]; V) \cap W^{1,2}((0, T); L^2(\Omega))$  which is a strong solution of problem (3.15) on  $(0, T)$ .*

*Proof.* Let  $u_0 \in D(\Phi)$  and  $f \in C^1([0, T]; V)$ . Let  $\lambda > 0$  and let  $u_\lambda$  be the unique strong solution of (3.18) which exists by Proposition 3.11. In the subsequent proofs,  $C > 0$  will always denote a constant that is independent of  $t, f, \lambda$ , which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. We multiply (3.18) by  $\frac{du_\lambda(t)}{dt}$  and we integrate the



resulting identity over  $(0, t)$ . Using (3.19), (3.20) and Proposition 3.3, we get that there exists a constant  $C > 0$  such that

$$\int_0^T \left\| \frac{du_\lambda}{dt}(t) \right\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0, T]} \Phi^\sigma(u_\lambda(t)) \leq C. \quad (3.35)$$

The estimates (3.20) and (3.35) imply that

$$\sup_{t \in [0, T]} \|u_\lambda(t)\|_V \leq C. \quad (3.36)$$

Furthermore, we also have there exists a constant  $C > 0$  such that

$$\sup_{t \in [0, T]} \|J_\lambda^{\bar{\psi}} u_\lambda(t)\|_V \leq C \quad (3.37)$$

on account of (3.22) and (3.35). Let now

$$g_\lambda := f(t) - \frac{du_\lambda}{dt}(t) + \partial_H \bar{\psi}_\lambda(u_\lambda(t)) \in \partial_H \bar{\Phi}^\sigma(u_\lambda(t)).$$

Then, passing to a subsequence of  $\{\lambda\}$  if necessary, we get that as  $\lambda \rightarrow 0^+$ ,

$$\begin{cases} u_\lambda \rightarrow u, & J_\lambda^{\bar{\psi}}(u_\lambda) \rightarrow u & \text{weakly star in } L^\infty((0, T); V), \\ u_\lambda \rightarrow u, & J_\lambda^{\bar{\psi}}(u_\lambda) \rightarrow u & \text{weakly in } W^{1,2}((0, T); L^2(\Omega)) \\ \partial_H \bar{\psi}_\lambda(u_\lambda(\cdot)) \rightarrow \partial_V \psi(u(\cdot)) & \text{strongly in } C([0, T]; L^{q'}(\Omega)), \\ g_\lambda \rightarrow g \in \partial_V \Phi^\sigma(u(\cdot)) & \text{weakly in } L^2((0, T); V^*). \end{cases} \quad (3.38)$$

The first two foregoing convergence properties follow from (3.35), (3.37) and (3.36). The third convergence property follows from (3.35) in light of Remark 3.7, part (c). On the other hand, the last convergence property follows from the second and third of (3.38) on the account of the fact that  $L^2(\Omega)$  and  $L^{q'}(\Omega)$  are both continuously embedded into  $V^*$ . Clearly, (3.38) also yields

$$g(t) = f(t) - \frac{du(t)}{dt} + \partial_V \psi(u(t)), \text{ a.e. } t \in (0, T).$$

Finally, since  $u_\lambda(t) \rightarrow u_0$  strongly in  $L^r(\Omega)$  as  $t \rightarrow 0^+$ , we may conclude that the limit function  $u$  is the unique strong solution to the auxiliary problem (3.15) on  $(0, T)$ . The proof of the proposition is finished.  $\square$

#### 4. PROOF OF THE MAIN RESULTS

In this section we prove the main results stated in Section 2.2.

**4.1. Proof of Theorem 2.3.** We can now complete the proof of the first main result of the article. This program will be divided into several steps.

**Step 1 (Additional uniform estimates).** We give further (uniform) estimates of solutions to the regularized problem (3.18) that will be needed in the sequel. Recall that  $p, q, r \in [2, \infty)$  satisfy  $p < q$  and (3.7). Let  $\lambda > 0$  and consider the unique strong solution  $u_\lambda$  to (3.18). Multiplying (3.18) by  $u_\lambda(t)$ , integrating the

resulting identity over  $(0, t)$  and using (3.20), we deduce

$$\begin{aligned}
& \frac{1}{2} \|u_\lambda(t)\|_{L^2(\Omega)}^2 + \int_0^t \overline{\Phi}^\sigma(u_\lambda(\tau)) d\tau \\
& \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + \int_0^t \|\partial_H \overline{\psi}(u_\lambda(\tau))\|_{L^{q'}(\Omega)} \|u_\lambda(\tau)\|_{L^q(\Omega)} d\tau \\
& \quad + \int_0^T \|f(\tau)\|_{V^*} \|u_\lambda(\tau)\|_V d\tau \\
& \leq \frac{1}{2} \|u_0\|_{L^2(\Omega)}^2 + C \int_0^t \psi(u_\lambda(\tau)) d\tau + C \int_0^T \|f(\tau)\|_{V^*}^{p'} d\tau \\
& \quad + \frac{1}{2} \int_0^t \overline{\Phi}^\sigma(u_\lambda(\tau)) d\tau + \frac{T}{2} \sigma^{\frac{p}{p'}}.
\end{aligned} \tag{4.1}$$

Lemma 3.6 together with (3.19) thus gives

$$\begin{aligned}
& \sup_{t \in [0, T]} \|u_\lambda(t)\|_{L^2(\Omega)}^2 + \int_0^T \overline{\Phi}^\sigma(u_\lambda(\tau)) d\tau \\
& \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + T\mathcal{F}(\sigma) + \int_0^T \|f(\tau)\|_{V^*}^{p'} d\tau \right).
\end{aligned}$$

Next, multiplying (3.18) by  $t \frac{du_\lambda(t)}{dt}$  and using the fact that

$$\begin{aligned}
\int_\Omega f(t) t \frac{du_\lambda(t)}{dt} dx &= \frac{d}{dt} \left( t \int_\Omega f(t) u_\lambda(t) dx \right) - \int_\Omega f(t) u_\lambda(t) dx \\
&\quad - t \int_\Omega \frac{df}{dt}(t) u_\lambda(t) dx,
\end{aligned}$$

we obtain

$$\begin{aligned}
& t \left\| \frac{du_\lambda(t)}{dt} \right\|_{L^2(\Omega)}^2 + \frac{d}{dt} \left[ t \overline{\Phi}^\sigma(u_\lambda(t)) \right] - \overline{\Phi}^\sigma(u_\lambda(t)) \\
& \leq \frac{d}{dt} \left[ t \overline{\psi}_\lambda(u_\lambda(t)) \right] - \overline{\psi}_\lambda(u_\lambda(t)) + \frac{d}{dt} \left( t \int_\Omega f(t) u_\lambda(t) dx \right) \\
& \quad - \int_\Omega f(t) u_\lambda(t) dx - t \int_\Omega \frac{df}{dt}(t) u_\lambda(t) dx.
\end{aligned}$$

Integrating the foregoing inequality over  $(0, t)$  and using (3.19) and (3.20) once more, we readily see that

$$\begin{aligned}
& \int_0^t \tau \left\| \frac{du_\lambda(\tau)}{d\tau} \right\|_{L^2(\Omega)}^2 d\tau + t \bar{\Phi}^\sigma(u_\lambda(t)) + \int_0^t \bar{\psi}_\lambda(u_\lambda(\tau)) d\tau \\
& \leq t \bar{\psi}_\lambda(u_\lambda(t)) + \int_0^t \bar{\Phi}^\sigma(u_\lambda(\tau)) d\tau + t \int_\Omega f(t) u_\lambda(t) dx \\
& \quad - \int_0^t \int_\Omega f(\tau) u_\lambda(\tau) dx d\tau - \int_0^t \tau \int_\Omega \frac{df}{d\tau}(t) u_\lambda(\tau) dx d\tau \\
& \leq \frac{t}{2} \bar{\Phi}^\sigma(u_\lambda(t)) + C \int_0^T \bar{\Phi}^\sigma(u_\lambda(\tau)) d\tau + C \sup_{\tau \in [0, T]} \tau \|f(\tau)\|_{V^*}^{p'} \\
& \quad + \frac{t}{4} \bar{\Phi}^\sigma(u_\lambda(t)) + C \int_0^t \|f(\tau)\|_{V^*}^{p'} d\tau + \int_0^T \left\| \tau \frac{df}{d\tau}(\tau) \right\|_{V^*}^{p'} d\tau + T \mathcal{F}(\sigma).
\end{aligned} \tag{4.2}$$

On the other hand, using the fact that

$$\sup_{t \in [0, T]} t \|f(t)\|_{V^*}^{p'} \leq C \left( \int_0^T \|f(t)\|_{V^*}^{p'} dt + \int_0^T \left\| t \frac{df}{dt}(t) \right\|_{V^*}^{p'} dt \right),$$

we further get from (4.2) that

$$\begin{aligned}
& \int_0^T t \left\| \frac{du_\lambda}{dt}(t) \right\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0, T]} t \bar{\Phi}^\sigma(u_\lambda(t)) \\
& \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + T \mathcal{F}(\sigma) + \int_0^T \|f(t)\|_{V^*}^{p'} dt + \int_0^T \left\| t \frac{df}{dt}(t) \right\|_{V^*}^{p'} dt \right).
\end{aligned} \tag{4.3}$$

Letting  $\lambda \rightarrow 0^+$ , from (3.38) and (4.1) we infer

$$\begin{aligned}
& \sup_{t \in [0, T]} \|u(t)\|_{L^2(\Omega)}^2 + \int_0^T \Phi^\sigma(u(\tau)) d\tau \\
& \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + T \mathcal{F}(\sigma) + \int_0^T \|f(t)\|_{V^*}^{p'} dt \right)
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
& \int_0^T t \left\| \frac{du}{dt}(t) \right\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0, T]} t \Phi^\sigma(u(t)) \\
& \leq C \left( \|u_0\|_{L^2(\Omega)}^2 + T \mathcal{F}(\sigma) + \int_0^T \|f(t)\|_{V^*}^{p'} dt + \int_0^T \left\| t \frac{df}{dt}(t) \right\|_{V^*}^{p'} dt \right),
\end{aligned} \tag{4.5}$$

for some constant  $C > 0$  independent of  $t$ ,  $f$  and  $\lambda > 0$ .

**Step 2 (Passage to limit).** Let  $T > 0$  be fixed,  $u_0 \in D(\Phi)$  and  $f \in C^1([0, T]; V)$ . Let  $\phi_\mu$  be the Moreau-Yosida approximation of  $\phi$  for  $\mu > 0$  and let  $u_\lambda$  ( $\lambda > 0$ ) be the unique strong solution to (3.18). Multiplying (3.18) by  $\partial_H \phi_\mu(u_\lambda(t))$  and using

the chain rule formula (see Proposition 3.3), we have

$$\begin{aligned} & \frac{d}{dt} \phi_\mu(u_\lambda(t)) + \int_{\Omega} g_\lambda(t) \partial_H \phi_\mu(u_\lambda(t)) dx \\ &= \int_{\Omega} \partial_H \bar{\psi}_\mu(u_\lambda(t)) \partial_H \phi_\mu(u_\lambda(t)) dx + \int_{\Omega} f(t) \partial_H \phi_\mu(u_\lambda(t)) dx. \end{aligned}$$

Let  $v_{\lambda,\mu}(t) := |J_\mu^\Phi u_\lambda(t)|^{\frac{r-2}{p}} J_\mu^\Phi u_\lambda(t)$  for a.e.  $t \in (0, T)$  and note that  $v_{\lambda,\mu}(t) \in W_0^{s,p}(\bar{\Omega})$  on account of Lemma 3.10. Inserting the estimates of Lemmas 3.10 and 3.8 into the foregoing identity, we readily have

$$\begin{aligned} & \frac{d}{dt} [\phi_\mu(u_\lambda(t))] + \frac{\beta C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,t) - v_{\lambda,\mu}(y,t)|^p}{|x-y|^{N+sp}} dx dy \\ & \leq \mathcal{F}(\sigma) \left[ \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,t) - v_{\lambda,\mu}(y,t)|^p}{|x-y|^{N+sp}} dx dy \right]^{1-\varepsilon} \\ & \quad + \int_{\Omega} f(t) \partial_H \phi_\mu(u_\lambda(t)) dx. \end{aligned} \quad (4.6)$$

Hölder's inequality and (3.21) allow us to deduce

$$\begin{aligned} \int_{\Omega} f(t) \partial_H \phi_\mu(u_\lambda(t)) dx & \leq \|f(t)\|_{L^r(\Omega)} \|\partial_H \phi_\mu(u_\lambda(t))\|_{L^{r'}(\Omega)} \\ & \leq C \|f(t)\|_{L^r(\Omega)} \phi(J_\lambda^\Phi(u_\lambda(t)))^{\frac{1}{r'}} \\ & \leq C \sigma^{\frac{1}{r'}} \|f(t)\|_{L^r(\Omega)}. \end{aligned} \quad (4.7)$$

Integrating (4.6) over  $(0, t)$ , using (4.7), and recalling that the function  $v_{\lambda,\mu} \in L^p((0, T); W_0^{s,p}(\bar{\Omega}))$ , there holds

$$\begin{aligned} & \phi_\mu(u_\lambda(t)) + \beta \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,\tau) - v_{\lambda,\mu}(y,\tau)|^p}{|x-y|^{N+sp}} dx dy d\tau \\ & \leq \phi_\mu(u_0) + \mathcal{F}(\sigma) \int_0^t \left[ \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x,\tau) - v_{\lambda,\mu}(y,\tau)|^p}{|x-y|^{N+sp}} dx dy \right]^{1-\varepsilon} d\tau \\ & \quad + C \sigma^{\frac{1}{r'}} \int_0^t \|f(\tau)\|_{L^r(\Omega)} d\tau, \end{aligned} \quad (4.8)$$

for some  $C > 0$  independent of  $\mu > 0$ . Therefore, since  $\phi_\mu(u_0) \leq \phi(u_0)$ , after passing to a subsequence of  $\{\mu\}$  if necessary, we can infer the existence of a function  $w_\lambda \in L^p((0, T); W_0^{s,p}(\bar{\Omega}))$  such that, as  $\mu \rightarrow 0^+$ ,

$$v_{\lambda,\mu} \rightarrow w_\lambda \text{ weakly in } L^p((0, T); W_0^{s,p}(\bar{\Omega})). \quad (4.9)$$

Next, it follows from Proposition 3.2 that

$$\frac{1}{2\mu} \|u_\lambda(t) - J_\mu^\Phi u_\lambda(t)\|_{L^2(\Omega)}^2 = \phi_\mu(u_\lambda(t)) - \phi(J_\mu^\Phi u_\lambda(t)) \leq \sigma. \quad (4.10)$$

Estimate (4.10) implies that  $J_\mu^\Phi u_\lambda(t) \rightarrow u_\lambda$  strongly in  $C([0, T]; L^2(\Omega))$ , as  $\mu \rightarrow 0^+$ . Hence, by (4.9) we can deduce that  $w_\lambda = v_\lambda = |u_\lambda|^{\frac{r-2}{p}} u_\lambda$ . Moreover, by lower

semi-continuity it follows that

$$\begin{aligned} & \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\lambda(x, \tau) - v_\lambda(y, \tau)|^p}{|x - y|^{N+sp}} dx dy d\tau \\ & \leq \liminf_{\mu \rightarrow 0^+} \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_{\lambda,\mu}(x, \tau) - v_{\lambda,\mu}(y, \tau)|^p}{|x - y|^{N+sp}} dx dy d\tau. \end{aligned}$$

Passing to the limit in (4.8) with respect to  $\mu \rightarrow 0^+$ , and applying Young's inequality, we get that

$$\begin{aligned} & \phi(u_\lambda(t)) + \frac{\beta}{2} \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_\lambda(x, \tau) - v_\lambda(y, \tau)|^p}{|x - y|^{N+sp}} dx dy d\tau \\ & \leq \phi(u_0) + t\mathcal{F}(\sigma) + t\frac{\beta}{2} + C\sigma^{\frac{1}{r}} \int_0^t \|f(\tau)\|_{L^r(\Omega)} d\tau, \end{aligned} \quad (4.11)$$

for all  $t \in [0, T]$ . Next, since the embedding  $V \hookrightarrow L^q(\Omega)$  is compact, the application of Ascoli's compactness lemma together with (3.38) yields

$$u_\lambda \rightarrow u \text{ strongly in } C([0, T]; L^q(\Omega)).$$

Since  $\phi(u_\lambda(t)) \leq \sigma$  for all  $t \in [0, T]$  and  $2(p+r-2)/p \leq r$ , then letting  $\lambda \rightarrow 0^+$  in (4.11), we also obtain that

$$\begin{cases} u_\lambda \rightarrow u & \text{weakly star in } L^\infty((0, T); L^r(\Omega)), \\ v_\lambda \rightarrow v & \text{weakly star in } L^\infty((0, T); L^2(\Omega)), \\ v_\lambda \rightarrow v & \text{weakly in } L^p((0, T); W_0^{s,p}(\overline{\Omega})), \end{cases} \quad (4.12)$$

where  $v_\lambda := |u_\lambda|^{\frac{r-2}{p}} u_\lambda$  and  $v := |u|^{\frac{r-2}{p}} u$ . We can then conclude from (4.11) and (4.12) that there exists a constant  $C > 0$  such that

$$\begin{aligned} & \phi(u(t)) + \frac{\beta}{2} \int_0^t \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x, \tau) - v(y, \tau)|^p}{|x - y|^{N+sp}} dx dy d\tau \\ & \leq \phi(u_0) + t\mathcal{F}(\sigma) + t\frac{\beta}{2} + C\sigma^{\frac{1}{r}} \int_0^t \|f(\tau)\|_{L^r(\Omega)} d\tau. \end{aligned} \quad (4.13)$$

The final estimate (4.13) implies that

$$\limsup_{t \rightarrow 0^+} \phi(u(t)) \leq \phi(u_0). \quad (4.14)$$

Since  $u \in C([0, T]; L^2(\Omega))$  and  $\phi$  is lower semi-continuous, we have that

$$\liminf_{t \rightarrow 0^+} \phi(u(t)) \geq \phi(u_0). \quad (4.15)$$

Since  $L^r(\Omega)$  is uniformly convex, we obtain from (4.14) and (4.15) that

$$u(t) \rightarrow u_0 \text{ strongly in } L^r(\Omega) \text{ as } t \rightarrow 0^+. \quad (4.16)$$

**Step 3 (Solution to the original problem)** Let  $T > 0$  be fixed,  $u_0 \in D(\Phi)$  and

$$f \in W^{1,p'}(0, T; W^{-s,p'}(\Omega) + L^{r'}(\Omega)) \cap L^{1+\gamma}(0, T; L^r(\Omega)) =: \mathcal{Y}_f \quad (4.17)$$

for some  $\gamma \geq 0$ . Let  $\mathcal{F} : [0, \infty) \rightarrow [0, \infty)$  be an increasing differentiable function satisfying the conclusion of Lemmas 3.6 and 3.8.

- If  $\gamma > 0$ , we take a non-increasing function  $T_\star : [0, \infty) \times [0, \infty) \rightarrow (0, T]$  independent of  $T$ ,  $u_0$  and  $f$  such that

$$T_\star(\eta, \xi) \left[ \mathcal{F}(\eta + 1) + \frac{\beta}{2} \right] + C(\eta + 1)^{\frac{1}{\sigma}} T_\star(\eta, \xi)^{\frac{\gamma}{\gamma+1}} \xi^{\frac{1}{1+\gamma}} \leq \frac{1}{2}.$$

- If  $\gamma = 0$ , we take a non-increasing function  $T_f : [0, \infty) \rightarrow (0, T]$  which depends on  $f$  but not on  $T$  and  $u_0$  such that

$$T_f(\eta) \left[ \mathcal{F}(\eta + 1) + \frac{\beta}{2} \right] + C(\eta + 1)^{\frac{1}{\sigma}} \int_0^{T_\star(\eta)} \|f(\tau)\|_{L^r(\Omega)} d\tau \leq \frac{1}{2}.$$

Let now

$$T_0 := T_\star \left( \phi(u_0), \int_0^T \|f(\tau)\|_{L^r(\Omega)}^{1+\gamma} d\tau \right) > 0 \text{ if } \gamma > 0$$

and

$$T_0 := T_f(\phi(u_0)) \text{ if } \gamma = 0.$$

Since  $\sigma = \phi(u_0) + 1$ , it follows that

$$\sup_{t \in [0, T_0]} \phi(u(t)) < \sigma.$$

Since  $\phi(u(t)) < \sigma$  for all  $t \in [0, T_0]$ , it follows from (3.14) that  $\partial_V \chi_{V_\sigma}(u(t)) = \{0\}$ , a.e.  $t \in [0, T_0]$ ; thus by (3.13),  $\partial_V \Phi^\sigma(u(t)) = \partial_V \Phi(u(t))$  for a.e.  $t \in [0, T_0]$ . We have shown that  $u$  is a strong solution of (3.8) on  $(0, T_0)$  and hence, a strong solution of (1.1) on  $(0, T_0)$  if the initial datum  $u_0 \in D(\Phi)$ .

**Step 4 (Final argument).** In this final step, we remove the assumption on the initial datum  $u_0 \in D(\Phi)$ . To this end, for fixed time  $T > 0$ , consider  $u_0 \in L^r(\Omega)$  and a function  $f$  satisfying (4.17). Let  $u_{0,n} \in D(\Phi)$  and  $f_n \in C^1([0, T]; V)$  be sequences such that  $u_{0,n} \rightarrow u_0$  strongly in  $L^r(\Omega)$  and  $f_n \rightarrow f$  strongly in  $\mathcal{Y}_f$ . Let  $\sigma := \phi(u_0) + 2$ . Then for sufficiently large  $n \geq n_0$ , we have

$$\phi(u_{0,n}) \leq \phi(u_0) + 1 \text{ and } \int_0^T \|f_n(t)\|_{L^r(\Omega)}^{1+\gamma} dt \leq \int_0^T \|f(t)\|_{L^r(\Omega)}^{1+\gamma} dt + 1.$$

Moreover, there exists a function  $h \in L^{1+\gamma}(0, T)$  such that, after passing to a subsequence if necessary, we also have

$$\|f_n(t)\|_{L^r(\Omega)} \leq h(t) \text{ for a.e. } t \in (0, T).$$

We now consider the  $n$ -approximate problem

$$\begin{cases} \frac{du_n}{dt}(t) + \partial_V \Phi(u_n(t)) - \partial_V \psi(u_n(t)) = f_n(t), & t \in (0, T) \\ u_n(0) = u_{0,n}. \end{cases} \quad (4.18)$$

Note that (4.18) possesses a strong solution  $u_n$  on  $(0, T_0)$  satisfying

$$\sup_{t \in [0, T_0]} \phi(u_n(t)) \leq \phi(u_{0,n}) + 1 \leq \sigma \quad (4.19)$$

for some  $T_0$  independent of  $n$ . Indeed, employing **Step 1** once again, we have the following alternatives.

- If  $\gamma > 0$ , then it is clear that

$$T_\star \left( \phi(u_{0,n}), \int_0^T \|f_n\|_{L^r(\Omega)}^{1+\gamma} dt \right) \geq T_\star \left( \Phi(u_0) + 1, \int_0^T \|f\|_{L^r(\Omega)}^{1+\gamma} dt + 1 \right).$$

- If  $\gamma = 0$ , since  $\|f_n(t)\|_{L^r(\Omega)} \leq h(t)$ , then we can choose the function  $T_h : [0, \infty) \rightarrow (0, T]$  such that

$$T_h(\eta) \left[ \mathcal{F}(\eta + 1) + \frac{\beta}{2} \right] + C(\eta + 1)^{\frac{1}{r'}} \int_0^{T_h(\eta)} |h(\tau)| d\tau \leq \frac{1}{2} \quad \text{and} \quad T_{f_n}(\eta) \geq T_h(\eta),$$

for any  $\eta \in [0, \infty)$ .

Hence, we can take  $T_0 > 0$  uniformly with respect to  $n$ . In the remainder of the proof,  $C > 0$  will denote a constant that is independent of  $t$ ,  $f$ ,  $n$ , and initial data, which only depends on the other structural parameters of the problem. Such a constant may vary even from line to line. It remains to derive uniform estimates for the solution  $u_n$  with respect to  $n$ . First, by estimates (4.4) and (4.5),

$$\sup_{t \in [0, T_0]} \|u_n(t)\|_{L^2(\Omega)} + \int_0^{T_0} \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x,t) - u_n(y,t)|^p}{|x-y|^{N+sp}} dx dy dt \leq C \quad (4.20)$$

and

$$\int_0^{T_0} t \left\| \frac{du_n}{dt}(t) \right\|_{L^2(\Omega)}^2 dt + \sup_{t \in [0, T_0]} \frac{t C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x,t) - u_n(y,t)|^p}{|x-y|^{N+sp}} dx dy \leq C. \quad (4.21)$$

Since  $\partial_V \Phi(u_n(t)) = (-\Delta)_p^s u_n(t)$  (see Proposition 3.5), then using (4.20) and the fact that (see the proof of Proposition 3.5)

$$\langle (-\Delta)_p^s u_n, u_n \rangle_{V^*, V} = \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x,t) - u_n(y,t)|^p}{|x-y|^{N+sp}} dx dy$$

we infer

$$\int_0^{T_0} \|\partial_V \Phi(u_n(t))\|_{W^{-s,p'}(\Omega)}^{p'} dt \leq C. \quad (4.22)$$

Application of Lemma 3.6 also yields

$$\begin{aligned} \int_0^{T_0} \|\partial_V \Phi(u_n(t))\|_{L^{q'}(\Omega)}^{q'} dt &\leq C \int_0^{T_0} \psi(u_n(t)) dt \\ &\leq \mathcal{F}(\sigma) \int_0^{T_0} (\Phi(u_n(t)) + 1)^{1-\varepsilon} dt \leq C. \end{aligned} \quad (4.23)$$

Therefore, since  $L^{q'}(\Omega) \hookrightarrow V^*$ , it follows from (4.18) that

$$\int_0^{T_0} \left\| \frac{du_n}{dt}(t) \right\|_{V^*}^{q'} dt \leq C. \quad (4.24)$$

Estimate (4.13) with  $u = u_n$ ,  $v = v_n = |u_n|^{\frac{r-2}{p}} u_n$ ,  $u_0 = u_{0,n}$  and  $f = f_n$  gives the uniform estimate

$$\int_0^{T_0} \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_n(x,t) - v_n(y,t)|^p}{|x-y|^{N+sp}} dx dy dt \leq C. \quad (4.25)$$

Since  $2(p+r-2)/r \leq r$ , it follows from (4.19) that the sequence  $v_n$  is bounded in  $L^\infty((0, T_0); L^2(\Omega))$ . We also notice the solution  $u_n$  satisfies the estimates (4.4) and

(4.5) with a constant  $C > 0$  independent of  $n$ . These uniform estimates allow us to pass to the limit, after a subsequence if necessary, such that as  $n \rightarrow \infty$ ,

$$\left\{ \begin{array}{ll} u_n \rightarrow u & \text{weakly star in } L^\infty((0, T_0); L^r(\Omega)), \\ u_n \rightarrow u & \text{weakly in } L^p((0, T_0); V), \\ t^{\frac{1}{p}} u_n \rightarrow t^{\frac{1}{p}} u & \text{weakly star in } L^\infty((0, T_0); W_0^{s,p}(\overline{\Omega})), \\ v_n \rightarrow v & \text{weakly star in } L^\infty((0, T_0); L^2(\Omega)), \\ v_n \rightarrow v & \text{weakly in } L^p((0, T_0); W_0^{s,p}(\overline{\Omega})), \\ \frac{du_n}{dt} \rightarrow \frac{du}{dt} & \text{weakly in } L^{q'}((0, T_0); V^*), \\ \sqrt{t} \frac{du_n}{dt} \rightarrow \sqrt{t} \frac{du}{dt} & \text{weakly in } L^2((0, T_0); L^2(\Omega)), \\ \partial_V \Phi(u_n(\cdot)) \rightarrow g(\cdot) & \text{weakly in } L^{p'}((0, T_0); W^{-s,p'}(\Omega)), \\ \partial_V \psi(u_n(\cdot)) \rightarrow h(\cdot) & \text{weakly in } L^{q'}((0, T_0); L^{q'}(\Omega)). \end{array} \right. \quad (4.26)$$

The first two of (4.26) follow from (4.19) and (4.20). The third and seventh convergence properties of (4.26) follow from (4.21). The fourth and fifth convergence properties are derived from the fact that  $v_n$  is bounded in  $L^\infty((0, T_0); L^2(\Omega))$  and from (4.24). The sixth convergence is an immediate consequence of (4.23), while the convergence  $\partial_V \Phi(u_n(\cdot)) \rightarrow g$  is a consequence of (4.22). Finally, the last of (4.26) follows from the sixth of (4.26) and  $\partial_V \Phi(u_n(\cdot)) \rightarrow g$ . Thus, we have shown

$$u \in C_w([0, T_0]; L^r(\Omega)) \cap C((0, T_0]; L^2(\Omega)).$$

We can now pass to strong convergence properties for the sequence  $u_n$ . Since the embeddings  $V \hookrightarrow L^q(\Omega)$  and  $L^r(\Omega) \hookrightarrow V^*$  are compact, it follows that

$$u_n \rightarrow u \text{ strongly in } L^p((0, T_0); L^q(\Omega)) \cap C([0, T_0]; V^*), \quad (4.27)$$

which together with (4.26) implies that  $v = |u|^{\frac{r-2}{p}} u$ . Moreover, it follows from (4.23) and (4.27) that  $u(t) \rightarrow u_0$  strongly in  $V^*$  as  $t \rightarrow 0^+$ .

It remains to show that  $\partial_V \psi(u(t)) = h(t)$  and  $g(t) = \partial_V \Phi(u(t))$  for a.e.  $t \in (0, T_0)$ . Indeed, if  $r < q$ , as in the proof of Lemma 3.6 we have using (3.12) that

$$\begin{aligned} & \int_0^T \|u_n(t) - u(t)\|_{L^q(\Omega)}^q dt \\ & \leq C \left( \int_0^{T_0} \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(u_n - u)(x, t) - (u_n - u)(y, t)|^p}{|x - y|^{N+sp}} dx dy dt \right)^{\frac{\alpha q}{p}} \\ & \quad \cdot \left( \int_0^{T_0} \|u_n(t) - u(t)\|_{L^r(\Omega)}^{(1-\alpha)q\nu} dt \right)^{\frac{1}{\nu}}, \end{aligned} \quad (4.28)$$

with  $\alpha > 0$  given by (3.11) and  $\nu = \frac{p}{p-\alpha q}$ . It follows from (4.19) and (4.27) that

$$u_n \rightarrow u \text{ strongly in } L^{(1-\alpha)q\nu}((0, T_0); L^r(\Omega)).$$

and, from (4.20) and (4.28), that

$$u_n \rightarrow u \text{ strongly in } L^q((0, T_0); L^q(\Omega)). \quad (4.29)$$

We notice that  $\partial_V \psi(u) = \partial_{L^q(\Omega)} \psi_{L^q}(u)$  if  $u \in V$ , where  $\psi_{L^q} : L^q(\Omega) \rightarrow [0, \infty)$  is defined by

$$\psi_{L^q}(u) := \frac{1}{q} \int_{\Omega} |u|^q dx, \quad \text{for all } u \in L^q(\Omega).$$



Since the subdifferential  $\partial_{L^q(\Omega)}\psi_{L^q}$  is demi-closed in  $L^q(\Omega) \times L^{q'}(\Omega)$ , we can apply [16, Proposition 1.1] to infer that  $h(t) = \partial_V \psi(u(t))$ , a.e.  $t \in (0, T_0)$ . If  $q \leq r$ , then (4.29) follows from (4.19) and (4.27). Hence, we have shown the first claim that  $h(t) = \partial_V \psi(u(t))$  for a.e.  $t \in (0, T_0)$ . In order to show that  $g(t) = \partial_V \Phi(u(t))$ , a.e.  $t \in (0, T_0)$ , we use (4.27) to take a set  $I \subset (0, T_0)$  such that  $u_n(\tau) \rightarrow u(\tau)$  strongly on  $L^q(\Omega)$  for all  $\tau \in I$  and  $|(0, T_0) \setminus I| = 0$ . Hence, for all  $\tau \in I$ ,

$$\begin{aligned} \int_{\tau}^{T_0} \langle \partial_V \Phi(u_n(t)), u_n(t) \rangle_{V^*, V} dt &= \int_{\tau}^{T_0} \langle f_n(t), u_n(t) \rangle_{V^*, V} dt \\ &\quad + \int_{\tau}^{T_0} \langle \partial_V \psi(u_n(t)), u_n(t) \rangle_{V^*, V} dt \\ &\quad - \frac{1}{2} \|u_n(T_0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u_n(\tau)\|_{L^2(\Omega)}^2. \end{aligned}$$

Since by (4.26),  $u \in W^{1,2}((\tau, T_0); L^2(\Omega))$ , then letting  $n \rightarrow \infty$  in the preceding equality, we deduce

$$\begin{aligned} \limsup_{n \rightarrow \infty} \int_{\tau}^{T_0} \langle \partial_V \Phi(u_n(t)), u_n(t) \rangle_{V^*, V} dt &\leq \int_{\tau}^{T_0} \langle f(t), u(t) \rangle_{V^*, V} dt \\ &\quad + \int_{\tau}^{T_0} \langle \partial_V \psi(u(t)), u(t) \rangle_{V^*, V} dt \\ &\quad - \frac{1}{2} \|u(T_0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|u(\tau)\|_{L^2(\Omega)}^2 \\ &= \int_{\tau}^{T_0} \langle g(t), u(t) \rangle_{V^*, V} dt. \end{aligned}$$

It follows from (4.26) that  $g(t) = \partial_V \Phi(u(t))$ , a.e.  $t \in (\tau, T_0)$ . Since  $\tau$  was arbitrary and  $|(0, T_0) \setminus I| = 0$ , we have that  $g(t) = \partial_V \Phi(u(t))$ , a.e.  $t \in (0, T_0)$ .

It remains to show that  $u(0) = u_0$  in the sense of  $L^r(\Omega)$ . Estimate (4.13) with  $u = u_n$ ,  $v = v_n$ ,  $u_0 = u_{0,n}$  and  $f = f_n$  allows us to pass to the limit as  $n \rightarrow \infty$ , to get

$$\phi(u(t)) \leq \phi(u_0) + t \left[ \mathcal{F}(\sigma) + \frac{\beta}{2} \right] + C\sigma^{\frac{1}{r}} \int_0^t \|f(\tau)\|_{L^r(\Omega)} d\tau,$$

for all  $t \in [0, T_0]$ . Arguing exactly as in (4.14)-(4.16) we easily find that  $u(t) \rightarrow u_0$  strongly in  $L^r(\Omega)$  as  $t \rightarrow 0^+$  and  $u \in C([0, T_0]; L^2(\Omega))$ . We have shown that  $u$  is a strong solution to problem (3.8) on  $(0, T_0)$  and hence, a strong solution to the initial-boundary value problem (1.1) on  $(0, T_0)$ . The proof of the theorem is complete.

**4.2. Proof of Theorem 2.5.** In this subsection we prove Theorem 2.5. We adapt a technique exploited by [18] to derive blow-up type results for the parabolic equation associated with the classical  $p$ -Laplace operator. We divide the proof into two parts. **Step 1 (Positive potential energies).** We first establish that there is a constant  $\beta > \alpha$  such that

$$\|u(t)\|_{W_0^{s,p}(\overline{\Omega})} \geq \beta \quad (4.30)$$

and

$$\|u(t)\|_{L^q(\Omega)} \geq C_* \beta \quad (4.31)$$

for all  $t \geq 0$  (and for as long as the strong solution exists). First, we notice that by definition (2.8) and the embedding (2.3), it holds

$$\begin{aligned} E(t) &\geq \frac{1}{p} \|u(t)\|_{W_0^{s,p}(\overline{\Omega})}^p - \frac{1}{q} C_*^q \|u(t)\|_{W_0^{s,p}(\overline{\Omega})}^q \\ &= \frac{1}{p} \tilde{x}^p - \frac{C_*^q}{q} \tilde{x}^q \stackrel{\text{def}}{=} h(\tilde{x}), \end{aligned} \quad (4.32)$$

where we have set  $\tilde{x} := \|u(t)\|_{W_0^{s,p}(\overline{\Omega})}$ . Clearly, the continuous function  $h$  is increasing on  $(0, \alpha)$  and decreasing on  $(\alpha, \infty)$  while  $h(\tilde{x}) \rightarrow -\infty$  as  $\tilde{x} \rightarrow \infty$  and  $h(\alpha) = E_0$ . Then, since  $E(0) < E_0$  it immediately follows that one has a constant  $\beta > \alpha$  such that  $h(\beta) = E(0)$ . On the other hand, setting  $\tilde{x}_0 = \|u_0\|_{W_0^{s,p}(\overline{\Omega})}$  then  $h(\tilde{x}_0) \leq E(0) = h(\beta)$  and  $\tilde{x}_0 \geq \beta$  on the account of (4.32). In order to show (4.30), we proceed to prove it by contradiction. To this end, let us assume that  $\|u(t_0)\|_{W_0^{s,p}(\overline{\Omega})} < \beta$  for some  $t_0 \in (0, T_0)$  on which the strong solution exists. By the continuity of this norm we can choose  $t_0 > 0$  such that  $\|u(t_0)\|_{W_0^{s,p}(\overline{\Omega})} > \alpha$ . By (4.32), we find that  $E(t_0) \geq h(\|u(t_0)\|_{W_0^{s,p}(\overline{\Omega})}) > h(\beta) = E(0)$  which contradicts the fact that  $E(t) \leq E(0)$ , for all  $t \in (0, T_0)$ , on which the strong solution exists, with the latter following easily by (2.10). Hence, we have proved (4.30). To prove (4.31), it remains to exploit (2.10) once again together with the definition of  $E(t)$  and (4.30) in order to see that

$$\frac{1}{q} \|u(t)\|_{L^q(\Omega)}^q \geq \frac{1}{p} \|u(t)\|_{W_0^{s,p}(\overline{\Omega})}^p - E(0) \geq \frac{1}{p} \beta^p - h(\beta) = \frac{C_*^q \beta^q}{q}$$

from which (4.31) follows. Next, defining  $H(t) := E_0 - E(t)$ , we have from [18, Lemma 4] that

$$0 < H(0) \leq H(t) \leq \frac{1}{q} \|u(t)\|_{L^q(\Omega)}^q, \quad (4.33)$$

provided that  $E(0) < E_0$  and  $\|u_0\|_{W_0^{s,p}(\overline{\Omega})} > \alpha$ , for as long as the strong solution exists.

**Step 2 (Blow-up in  $L^2$ -norm).** As in [18] (and references therein), setting  $G(t) := \frac{1}{2} \|u(t)\|_{L^2(\Omega)}^2$ , we have

$$\begin{aligned} G'(t) &= \int_{\Omega} u(t) \partial_t u(t) dx = \int_{\Omega} |u(t)|^q dx - \|u(t)\|_{W_0^{s,p}(\overline{\Omega})}^p \\ &= (1-p) \int_{\Omega} |u(t)|^q dx - pH(t) - pE_0, \end{aligned}$$

owing to the definition of  $H$ . By Step 1, (4.30)-(4.31), it is easy to check that

$$pE_0 = \frac{\alpha^q (q-p)}{\beta^q q} C_*^q \beta^q \leq \frac{\alpha^q (q-p)}{\beta^q} \|u(t)\|_{L^q(\Omega)}^q, \quad (4.34)$$

for as long as the strong solution exists. Setting  $d = (1 - \alpha^q/\beta^q)(q-p) > 0$ , it follows that

$$G'(t) \geq d \|u(t)\|_{L^q(\Omega)}^q + pH(t) \geq 0. \quad (4.35)$$

On the other hand, by Hölder's inequality we observe that

$$G^{\frac{q}{2}}(t) \leq L_{q,\Omega} \|u(t)\|_{L^q(\Omega)}^q, \quad L_{q,\Omega} := \left(\frac{1}{2}\right)^{\frac{q}{2}} |\Omega|^{\frac{q}{2}-1}. \quad (4.36)$$

Thus, combining (4.35)-(4.36), we get  $G'(t) \geq (d/L_{q,\Omega}) G^{q/2}(t)$  and one can directly integrate this inequality over  $(0, t)$ ,  $t > 0$ . It follows that

$$G^{\frac{q}{2}-1}(t) \geq \left( G^{1-q/2}(0) - \frac{d}{d/L_{q,\Omega}} \left( \frac{q}{2} - 1 \right) t \right)^{-1}$$

which shows that  $G(t)$  blows-up in finite time with a time  $t \leq t_*$ , given by (2.11). The proof is finished.

## 5. APPENDIX

We now prove Proposition 3.5, Lemma 3.9, Lemma 3.10 and Proposition 2.4.

**Proof of Proposition 3.5.** Let  $f \in V^*$  and  $u \in V = W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$ . We claim that  $f = \partial_V \Phi(u)$  if and only if for every  $v \in V$ ,

$$\begin{aligned} & \langle f, v \rangle_{V^*, V} \\ &= \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} dx dy. \end{aligned} \quad (5.1)$$

Indeed, let  $f \in V^*$  and  $u \in V = W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$  be such that (5.1) holds for every  $v \in V$ . Then (5.1) holds with  $v$  replaced by  $v - u$ . Using the following well-known inequality

$$\frac{|b|^p}{p} - \frac{|a|^p}{p} \geq |a|^{p-2} a(b - a), \text{ for any } a, b \in \mathbb{R},$$

we get that for every  $v \in V$ ,

$$\begin{aligned} & \Phi(v) - \Phi(u) \\ &= \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v(x) - v(y)|^p - |u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \\ &\geq \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))((v - u)(x) - (v - u)(y))}{|x - y|^{N+sp}} dx dy \\ &= \langle f, v - u \rangle_{V^*, V}. \end{aligned}$$

Hence,  $f = \partial_V \Phi(u)$ . Conversely, let  $u \in V$  and set  $f := \partial_V \Phi(u) \in V^*$ . Then by definition, for every  $v \in V$ , we have that

$$\Phi(v) - \Phi(u) \geq \langle f, v - u \rangle_{V^*, V}. \quad (5.2)$$

Let  $t \in [0, 1]$ ,  $w \in V$  and set  $v = tw + (1 - t)u$  in (5.2). Then

$$\begin{aligned} & t \langle f, w - u \rangle_{V^*, V} \\ &\leq \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|(tw + (1 - t)u)(x) - (tw + (1 - t)u)(y)|^p - |u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy. \end{aligned} \quad (5.3)$$

Using the Dominated Convergence Theorem, we get from (5.3) that

$$\begin{aligned} & \langle f, w - u \rangle_{V^*, V} \leq \lim_{t \downarrow 0} \frac{\Phi(tw + (1 - t)u) - \Phi(u)}{t} \\ &= \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))((w - u)(x) - (w - u)(y))}{|x - y|^{N+sp}} dx dy. \end{aligned} \quad (5.4)$$

Replacing  $w$  by  $u + w$  in (5.4), we get that for every  $w \in V$ ,

$$\begin{aligned} & \langle f, w \rangle_{V^*, V} \\ & \leq \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+sp}} dx dy. \end{aligned} \quad (5.5)$$

Since (5.5) holds with  $w$  replaced by  $-w$ , it follows that

$$\langle f, w \rangle_{V^*, V} = \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)|^{p-2} \frac{(u(x) - u(y))(w(x) - w(y))}{|x - y|^{N+sp}} dx dy$$

and we have shown (5.1). The proof of the claim is finished.  $\square$

**Proof of Lemma 3.9.** We prove the inequality by elementary analysis. Let the function  $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be given by

$$g(z, t) = |z - t|^{p-2} (z - t) \left( |z|^{r-2} z - |t|^{r-2} t \right) - C_{r,p} \left| |z|^{\frac{r-2}{p}} z - |t|^{\frac{r-2}{p}} t \right|^p, \quad (5.6)$$

where we recall that

$$C_{r,p} = (r-1) \left( \frac{p}{r+p-2} \right)^p.$$

Using the definition of  $\mathcal{E}$ , we first notice that (3.28) is equivalent to showing that

$$g(z, t) \geq 0, \quad \forall (z, t) \in \mathbb{R}^2. \quad (5.7)$$

Second, we mention that it is easy to verify that

$$g(z, t) = g(t, z), \quad g(z, 0) \geq 0, \quad g(0, t) \geq 0 \quad \text{and} \quad g(z, t) = g(-z, -t).$$

Therefore, without any restriction, we may assume that  $z \geq t$  and hence, we have that

$$g(z, t) = (z - t)^{p-1} \left( |z|^{r-2} z - |t|^{r-2} t \right) - C_{r,p} \left| |z|^{\frac{r-2}{p}} z - |t|^{\frac{r-2}{p}} t \right|^p,$$

A simple calculation shows that

$$\frac{p}{r+p-2} \left[ |z|^{\frac{r-2}{p}} z - |t|^{\frac{r-2}{p}} t \right] = \int_t^z |\tau|^{\frac{r-2}{p}} d\tau. \quad (5.8)$$

Since the function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  given by  $\varphi(\tau) = |\tau|^p$  ( $p > 1$ ) is convex, then using the well-known Jensen inequality, it follows from (5.8) that

$$\begin{aligned} C_{r,p} \left| |z|^{\frac{r-2}{p}} z - |t|^{\frac{r-2}{p}} t \right|^p &= (r-1) \left| \frac{p}{r+p-2} \left[ |z|^{\frac{r-2}{p}} z - |t|^{\frac{r-2}{p}} t \right] \right|^p \\ &= (r-1) \left| \int_t^z |\tau|^{\frac{r-2}{p}} d\tau \right|^p \\ &= (r-1)(z-t)^p \left| \int_t^z |\tau|^{\frac{r-2}{p}} \frac{d\tau}{z-t} \right|^p \\ &\leq (r-1)(z-t)^p \int_t^z |\tau|^{r-2} \frac{d\tau}{z-t} \\ &= (r-1)(z-t)^{p-1} \int_t^z |\tau|^{r-2} d\tau \\ &= (z-t)^{p-1} \left( |z|^{r-2} z - |t|^{r-2} t \right). \end{aligned}$$

We have shown (5.7) and this completes the proof of lemma.  $\square$

**Proof of Lemma 3.10.** Let  $u \in D(\partial_V \Phi) = V$  and  $\mu > 0$ . It follows from Proposition 3.2 that  $J_\mu^\phi u \in V = D(\partial_V \Phi)$ . Therefore,

$$\partial_H \phi_\mu(u) = \frac{u - J_\mu^\phi u}{\mu} \in V, \text{ for all } u \in V.$$

Next, let  $w \in D(\partial_V \Phi) \cap D(\partial_H \phi) = V$  be such that  $\partial_H \phi(w) \in V$ . Since  $\partial_H \phi(w) = |w|^{r-2} w \in W_0^{s,p}(\overline{\Omega})$ , it follows from Lemma 3.9 that there exists a constant  $\beta = \beta(r, p, s) > 0$  such that

$$\begin{aligned} & \frac{\beta C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)|^{\frac{r-2}{p}} w(x) - |w(y)|^{\frac{r-2}{p}} w(y)|^p}{|x-y|^{N+sp}} dx dy \\ & \leq \frac{C_{N,p,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |w(x) - w(y)|^{p-2} \frac{(w(x) - w(y))(|w(x)|^{r-2} w(x) - |w(y)|^{r-2} w(y))}{|x-y|^{N+sp}} dx dy \\ & = \int_{\Omega} (-\Delta)_p^s w |w|^{r-2} w dx = \int_{\mathbb{R}^N} (-\Delta)_p^s w |w|^{r-2} w dx = \langle \partial_V \Phi(w), \partial_H \phi(w) \rangle_{V^*, V}. \end{aligned} \quad (5.9)$$

Now, let  $v_\mu := |J_\mu^\phi u|^{\frac{r-2}{p}} J_\mu^\phi u$ . Note that  $v_\mu = 0$  on  $\mathbb{R}^N \setminus \Omega$  and using that  $\partial_H \phi(J_\mu^\phi u) = \partial_H \phi_\mu(u) \in V$ , we get that

$$\begin{aligned} \int_{\mathbb{R}^N} |v_\mu|^p dx &= \int_{\Omega} |v_\mu|^p dx = \int_{\Omega} |J_\mu^\phi u|^{r-2+p} dx \leq \int_{\Omega} |u|^{r-2+p} dx \\ &\leq \int_{\Omega} |u|^{r-1} |u|^{p-1} dx \leq \left( \int_{\Omega} |u|^{p(r-1)} dx \right)^{\frac{1}{p}} \left( \int_{\Omega} |u|^p dx \right)^{\frac{p-1}{p}} < \infty. \end{aligned}$$

Since  $J_\mu^\phi u \in D(\partial_V \Phi) = V$  and  $\partial_H \phi(J_\mu^\phi u) = \partial_H \phi_\mu(u) \in V$ , (5.9) allows us to deduce that  $v_\mu \in W_0^{s,p}(\overline{\Omega})$ . We have shown the first claim (3.29).

Finally, assume that  $u \in D(\partial_V \Phi^\sigma) = V_\sigma$ . Then by Propositions 3.2 and 3.4, there holds  $J_\mu^\phi u \in V_\sigma$ . Hence, for all  $u \in D(\partial_V \Phi^\sigma) = V_\sigma$  and  $g \in \partial_V \Phi^\sigma(u)$ , we have the estimate

$$\begin{aligned} \langle g, \partial_H \phi_\lambda(u) \rangle_{V^*, V} &\geq \lambda^{-1} \left( \Phi^\sigma(u) - \Phi^\sigma(J_\lambda^\phi(u)) \right) \\ &= \lambda^{-1} \left[ \Phi(u) - \Phi(J_\lambda^\phi(u)) \right] \\ &\geq \lambda^{-1} \langle \partial_V \Phi(J_\lambda^\phi(u)), u - J_\lambda^\phi(u) \rangle_{V^*, V} \\ &= \lambda^{-1} \langle \partial_V \Phi(J_\lambda^\phi(u)), \partial_H \phi_\lambda(u) \rangle_{V^*, V}. \end{aligned} \quad (5.10)$$

Combining (5.9) together with (5.10), we easily obtain (3.29). This finishes the proof of lemma.  $\square$

**Proof of Proposition 2.4.** As in Step 4 of the proof of Theorem 2.3, we can pick a sufficiently smooth sequence of initial data  $u_{0,n} \in D(\Phi) \cap V$  such that  $u_{0,n} \rightarrow u_0$  strongly in  $V = W_0^{s,p}(\overline{\Omega}) \cap L^r(\Omega)$  as  $n \rightarrow \infty$ . Then we consider again the approximate problem (4.18) on  $(0, T_0)$  (of course, now  $f_n \equiv 0$ ) which we test it again with  $\partial_t u_n \in L^2((0, T_0); L^2(\Omega))$ . We note that every smooth solution of the approximate problem (4.18) does possess such regularity. We obtain

$$\frac{d}{dt} E_n(t) + \|\partial_t u_n(t)\|_{L^2(\Omega)}^2 = 0, \quad (5.11)$$

for all  $t \in (0, T_0)$ , and where we have set

$$E_n(t) := \frac{C_{N,p,s}}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_n(x,t) - u_n(y,t)|^p}{|x-y|^{N+sp}} dx dy - \frac{1}{q} \int_{\Omega} |u_n(x,t)|^q dx.$$

Integrating (5.11) over the interval  $(0, t)$  allows us to deduce

$$E_n(t) \leq E_n(0)$$

for all  $t \in (0, T_0)$ . We can now easily conclude the proof of Proposition 2.4 exploiting the foregoing inequality. Indeed, recalling that  $r > \frac{N(q-p)}{sp}$  with  $q > p$ , we see that  $u_n(t) \rightarrow u(t)$  strongly in  $L^q(\Omega)$ , a.e. for  $t \in (0, T_0)$ , owing to (4.29). Passing to the limit as  $n \rightarrow \infty$ , we first have  $E_n(0) \rightarrow E(0)$  and then also

$$\int_{\Omega} |u_n(x,t)|^q dx \rightarrow \int_{\Omega} |u(x,t)|^q dx \text{ a.e. for } t \in (0, T_0).$$

This basic fact together with the weak lower-semicontinuity of the  $W_0^{s,p}(\overline{\Omega})$ -norm entails that  $E(t) \leq \liminf_{n \rightarrow \infty} E_n(t)$  and this concludes the proof of (2.10).  $\square$

#### REFERENCES

- [1] D.R. Adams and L.I. Hedberg, *Function Spaces and Potential Theory*. Grundlehren der Mathematischen Wissenschaften **314**. Springer-Verlag, Berlin, 1996.
- [2] G. Akagi, *Local existence of solutions to some degenerate parabolic equation associated with the  $p$ -Laplacian*. J. Differential Equations **241** (2007), 359–385.
- [3] G. Akagi and M. Otani, *Evolution inclusions governed by subdifferentials in reflexive Banach spaces*. J. Evol. Equ. **4** (2004), 519–541.
- [4] G. Akagi and M. Otani, *Evolution inclusions governed by the difference of two subdifferentials in reflexive Banach spaces*. J. Differential Equations **209** (2005), 392–415.
- [5] C. Bjorland, L. Caffarelli and A. Figalli, *Nonlocal tug-of-war and the infinity fractional Laplacian*. Comm. Pure Appl. Math. **65** (2012), 337–380.
- [6] H. Brézis, *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.
- [7] H. Brezis, M.G. Crandall and A. Pazy, *Perturbations of nonlinear maximal monotone sets in Banach space*. Comm. Pure Appl. Math. **23** (1970), 123–144.
- [8] L. Caffarelli, J.-M. Roquejoffre and Y. Sire, *Variational problems for free boundaries for the fractional Laplacian*. J. Eur. Math. Soc. **12** (2010), 1151–1179.
- [9] L. Caffarelli, S. Salsa and L. Silvestre, *Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian*. Invent. Math. **171** (2008), 425–461.
- [10] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*. Comm. Partial Differential Equations **32** (2007), 1245–1260.
- [11] E. Di Nezza, G. Palatucci and E. Valdinoci, *Hitchhiker's Guide to the Fractional Sobolev Spaces*. Bull. Sci. Math. **136** (2012), 521–573.
- [12] A. Fiscella, R. Servadei and E. Valdinoci, *Density properties for fractional Sobolev spaces*. Ann. Acad. Sci. Fenn. Math. **40** (2015), 235–253.
- [13] C.G. Gal, M. Warma, *Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions*. Discrete Contin. Dyn. Syst. **36** (2016), 1279–1319.
- [14] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Monographs and Studies in Mathematics, **24**. Pitman, Boston, MA, 1985.
- [15] A. Jonsson and H. Wallin, *Function Spaces on Subsets of  $\mathbb{R}^N$* . Math. Rep. **2** (1984).
- [16] N. Kenmochi, *Some nonlinear parabolic variational inequalities*. Israel J. Math. **22** (1975), 304–331.
- [17] J.-L. Lions and E. Magenes, *Non-homogeneous Boundary Value Problems and Applications*. Vol. I. Springer-Verlag, New York-Heidelberg, 1972.

- [18] W. Liu, M. Wang, *Blow-up of the solution for a  $p$ -Laplacian equation with positive initial energy*. Acta Appl. Math. **103** (2008), 141–146.
- [19] R. E. Showalter, *Monotone Operators in Banach Space and Nonlinear Partial Differential Equations*. Amer. Math. Soc., Providence, RI, 1997.
- [20] D. W. Stroock, *An introduction to the theory of large deviations*. Universitext, Springer-Verlag, New York, 1984.
- [21] N. Th. Varopoulos, *Hardy-Littlewood theory for semigroups*. J. Funct. Anal. **63** (1985), 240–260.
- [22] L. Tang, *Random homogenization of  $p$ -Laplacian with obstacles in perforated domain and related topics*. *Ph.D Dissertation, The University of Texas at Austin*, 2011.
- [23] M. Warma, *The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets*. Potential Anal. **42** (2015), 499–547.
- [24] M. Warma, *The fractional Neumann and Robin type boundary conditions for the regional fractional  $p$ -Laplacian*. NoDEA Nonlinear Differential Equations Appl. **23** (2016), no. 1, Art. 1, 46 pp.
- [25] M. Warma, *Local Lipschitz continuity of the inverse of the fractional  $p$ -Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains*. Nonlinear Anal. **135** (2016), 129–157.

C. G. GAL, DEPARTMENT OF MATHEMATICS, FLORIDA INTERNATIONAL UNIVERSITY, MIAMI, 33199 (USA)

*E-mail address:* cgal@fiu.edu

M. WARMA, UNIVERSITY OF PUERTO RICO, FACULTY OF NATURAL SCIENCES, DEPARTMENT OF MATHEMATICS (RIO PIEDRAS CAMPUS), PO BOX 70377 SAN JUAN PR 00936-8377 (USA)

*E-mail address:* mahamadi.warma1@upr.edu, mjwarma@gmail.com